Problems for M.Sc. Workshop no.11, January 6, 2013
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Definition Let $N_\delta(F)$ be the smallest number of sets of diameter at most δ which can cover a set F. The limit

$$\limsup_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta}$$

is called the upper box (or Minkowski) dimension of F and is denoted by $\overline{\dim}_B F$. Replacing lim sup by lim inf here one obtains the lower box (or Minkowski) dimension of F denoted by $\underline{\dim}_B F$. If lim sup = lim inf above then their common value is called the box (or Minkowski) dimension of F and is denoted $\dim_B F$.

68. Prove that if F is the standard Cantor set then $\dim_B F = \frac{\log 2}{\log 3}$.

Compute the box dimensions of the sets F described in problems 69–71 below.

69. Let $x = \sum_{i=1}^{\infty} x_i(x)2^{-i}$, $x_i = 0$ or $= 1$ be diadic (binary) expansions of numbers $x \in [0,1]$ which is unique if we do not allow expansions with $x_i = 1$ for all sufficiently large i's. Let $A = (a_{ij})$, $i,j = 0,1$ be 2×2 matrix so that $a_{00} = 0$, $a_{01} = a_{10} = a_{11} = 1$. Define $F = \{x \in [0,1] : a_{x_i(x),x_{i+1}(x)} = 1\,\forall i \geq 1\}$.

70. The set F consists of points $(x,y) \in [0,1]^2$ such that the decimal expansions of neither x nor y contain the digit 5.

71. The set F, called the Sierpinski gasket, is constructed in the following way. We start with an equilateral triangle, partition it into 4 equilateral triangles of half the side and keep 3 of them having one common vertix with the initial triangle and throw away the triangle in the middle. Continue the same procedure with the remaining triangles and what is left after infinitely many steps is our set F.