Problems for M.Sc. Workshop no.10, December 30, 2012

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61. Let A and B be 1979×1979 square real matrices such that $AB = 0$. Prove that either $\det(A + A^*) = 0$ or $\det(B + B^*) = 0$.

62. What can be the maximal dimension of a linear space consisting of $n \times n$ matrices with zero determinant.

63. Suppose that the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analitic and one-to-one in the disc $D = \{z : |z| < 1\}$. Prove that $\mathrm{Leb}(f(D)) \geq \pi$ (Leb denotes the Lebesgue measure) and the equality holds if and only if $f(z) = z$.

64. Prove that any rational number x such that the limit $\lim_{n\to\infty}\{x^n\}$ exists (where $\{a\}$ denotes the fractional part of a) is an integer or $|x| < 1$.

65. Infinely many convex sets of Lebesgue measure $\frac{1}{10}$ are contained in a unit square. Prove that for any $\varepsilon > 0$ we can choose two of these sets whose intersection has Lebesgue measure at least $\frac{1}{10} - \varepsilon$. Show that without the convexity assumption the claim is not true, in general.

66. Let f be a bounded measurable function on $\mathbb{R}$ such that $\mathrm{Leb}\{x : f(x) > 0\} > 0$ and $\mathrm{Leb}\{x : f(x) < 0\} > 0$ where Leb denotes the Lebesgue measure. Prove that there exists a Borel probability measure μ on $\mathbb{R}$ equivalent to Leb (i.e. they have the same sets of zero measure) such that $\int f d\mu = 0$.

67.Let f be a bounded measurable function on $\mathbb{R}$. Let μ be a Borel probability measure on $\mathbb{R}$ such that $\int f d\mu = 0$ and for any other Borel probability measure ν on $\mathbb{R}$ which is equivalent to μ $\int f d\nu \neq 0$. Prove that the support of μ contains at most two points.