Problems for M.Sc. Workshop no.1, October 21, 2012

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1. Prove that for any bounded continuous function g and each $a > 0$ the limit

$$f_a(\lambda) = \lim_{n\to\infty} e^{-\lambda n} \sum_{0\le k\le an} g(\frac{k}{n})\frac{(\lambda n)^k}{k!}$$

exists and $f_a(\lambda) = 0$ if $a < \lambda$, $f_a(\lambda) = \frac{1}{2}g(\lambda)$ if $a = \lambda$, and $f_a(\lambda) = g(\lambda)$ if $a > \lambda$.

2. Suppose someone is choosing two real numbers a and b independently at random according to an unknown continuous distribution on $\mathbb{R}$ and tells you a. You have to guess whether a is smaller or larger than b. Is there a strategy which will give you a success with probability larger than $\frac{1}{2}$.

3. For any set $A \subset \mathbb{R}^d$ denote $D(A) = \{x - y : x, y \in A\}$. Prove that if A is the standard Cantor set on $[0, 1]$ then $D(A) = [-1, 1]$. Let A be the subset of $[0, 1]$ consisting of all numbers whose decimal expansion contains only: a) odd digits, b) only digits 1,2,3,4. Check whether $D(A) = [-1, 1]$ in these two cases.

4. Prove that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ cannot be an integer for $n > 1$.

5. Prove that any rational number $r \in (0, 1]$ is a sum of finite number of reciprocals of different natural numbers (i.e. $r = \sum_{i=1}^{k} 1/n_i$, $n_i \ne n_j$ if $i \ne j$ and $n_i \in \mathbb{N}\ \forall i$).

6. Prove that n robbers can divide among themselves their booty which is arbitrarily divisible so that every one of them would feel that he got a fair part. Proof for $n = 2 :$ A asks B to divide the booty into 2 equal (from B's point of view) parts and then A chooses the part he likes best. In this way B feels satisfied since he considers both parts equal and A is satisfied since he chose the part which in his opinion is, at least, not less than the other one, i.e. that he got, at least, half of the booty.