Definition 0.1. Given two representations $(\pi_1, V_1), (\pi_2, V_2)$ of a group G we denote by $Hom_G(V_1, V_2)$ the linear space of linear maps $T : V_1 \to V_2$ such that $T \circ \pi_1(g) = \pi_2(g) \circ T$ for all $g \in G$.

Problems.

1) Prove *easy to show* on p.8 in [S] (Serre's book).

2) Let $(\pi_1, V_1), (\pi_2, V_2)$ be non-equivalent irreducible representations of a group $G, (\pi, V) := (\pi_1, V_1) \oplus (\pi_2, V_2)$. Show that any non-trivial invariant subspace of V is either equal to V_1 or is equal to V_2.

3) Construct a 2-dimensional representations of the group $\mathbb{Z}$ which is reducible but is not equivalent to a direct sum of irreducible representations.

4) Let G be a finite group, X, Y be finite G-sets. Show that the dimension of the space $Hom_G(\mathbb{C}[X], \mathbb{C}[Y])$ is equal to the number of G-orbits on $X \times Y$ under the diagonal action of $G, g : (x, y) \to (gx, gy)$.

5) Let S_n be the group of permutations of the set I with n elements [the symmetric group in n symbols]. For any $r, 0 \leq r \leq n$ we denote by X_r the set of subsets of I with r elements.

The group S_n acts naturally on X_r for each $r, 0 \leq r \leq n$ and we denote by ρ_r the permutation representation of the group S_n on the space $\mathbb{C}[X_r]$. Assume that $n > 6$.

a) Find $dim_{\mathbb{C}}(\mathbb{C}[X_r])$.

b) Decompose representations ρ_0, ρ_1, ρ_2 into a sum of irreducible representations and find dimensions of these irreducible representations.

c) Do the same for $r = 3$.