We assume that $char(k) \neq 2$ and k is algebraically closed.

We will introduce notations
$A_n = sl_{n+1}$,
$B_n = so_Q$ when Q is a non-degenerate quadratic form on a vector space V of dimension $2n+1$
$C_n = sp_B$ when B is a non-degenerate symplectic form on a vector space $V, dim(V) = 2n$ and
$D_n = so_Q$ when Q is a non-degenerate quadratic form on a vector space V of dimension $2n$ [see Lecture 1].

The Lie algebras A_n, B_n, C_n, D_n are called the *classical* Lie algebras. Since classical Lie algebras are presented as subalgebras of $End(V)$ they have the *standard* representation on V.

Let $<,>$ be the symplectic form on k^{2n} such that $< v_i, v_{2n-i+1} >= 1$ for $1 \leq i \leq n, < v_i, v_{2n-i+1} >= -1$ for $n+1 \leq i \leq 2n$ and $< v_i, v_j >= 0$ if $i + j \neq 2n+1$. Let $\mathfrak{g} := sp_{<,>}$.

Problem 0.1. *a) If we write $2n \times 2n$ matrix $x \in End(k^{2n})$ in the form*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are $n \times n$ matrices then $x \in \mathfrak{g}$ iff

$D = -^t A, B =^t B$ and $C =^t C$ where $^t A$ is the transpose of A in repect of the second diagonal.

b) The the subset $\mathfrak{h}$ of diagonal matrices in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$.

c) $\mathfrak{h}$ consists of diagonal matrices $\{\lambda_i\}, 1 \leq i \leq 2n$ such that $\lambda_{2n-i} = -\lambda_i$.

d) The set R of roots in $\mathfrak{g}$ in respect to $\mathfrak{h}$ is a union

$$R = R' \cup R'', R' := \{e_i - e_j\}, R'' := \{\pm(e_i + e_j)\} 1 \leq i, j \leq n, i1 \neq jj$$

where $e_i \in \mathfrak{h}^\vee, 1 \leq i \leq n$ are linear functionals on $\mathfrak{h}$ such that $e_{i_0}(\{\lambda_i\}) := \lambda_{i_0}$.

Problem 0.2. *a) Show that*

a) The Lie algebras A_n, B_n, C_n and D_n are reductive for all $n > 0$.

b) The Lie algebras A_n, B_n and C_n are simple for all $n > 0$ and $A_1 = B_1 = C_1$.

c) $D_1 = k, D_2 = A_1 \times A_1$ and the Lie algebras D_n are simple for $n > 2$.

d)* Construct isomorphisms $D_3 = A_3$ and $B_2 = C_2$.

A hint. Consider the exterior square $\Lambda^2(V)$ where V is the standard representation of A_3 and C_2.

For any Lie algebra $\mathfrak{g} \subset End(V)$ and $v \in V$ we define

$$\mathfrak{g}_v := \{x \in \mathfrak{g} | xv \in kv\}$$

e) Find $r_v := r(\mathfrak{g}_v) \subset \mathfrak{g}_v, s_v := \mathfrak{g}_v / r(\mathfrak{g}_v)$ and construct explicitly the imbedding $s_v \hookrightarrow \mathfrak{g}_v$ for standard representations of classical Lie algebras.

A hint. In the case when $\mathfrak{g} = B_n$ or $\mathfrak{g} = D_n$ one should consider separately the case when $Q(v) = 0$ and the case when $Q(v) \neq 0$.

f) Problems 6.5 and 6.6.

g) Find a Cartan subalgebras for Lie algebras $\mathfrak{g} = sl_n, so_Q$ and construct the root decomposition.

A hint. Choose convenient bilinear and quadratic forms. The easiest quadratic forms to use are $Q(x_1, ..., x_{2n}) = \sum_{i=1}^n x_i x_{2n-i+1}$ and $Q(x_1, ..., x_{2n+1}) = \sum_{i=1}^n x_i x_{2n-i+2} + x_{n+1}^2$

In the next problem we have $k = \mathbb{R}$. So k is not algebraically closed.

h) Find a Cartan subalgebra for Lie algebras so_Q in the case when $k = \mathbb{R}, Q(x_1, ..., x_n) = \sum_{i=1}^n x_i^2$.