Definition 0.1. *Let k be a field and V a k-vector space.*

a) A subset $M \subset End(V)$ is nilpotent of level N if $A_1...A_N = 0$ for any $A_1...A_N \in M$. M is nilpotent if it is nilpotent of level N for some N

b) A subset $M \subset End(V)$ is weakly closed if $[M, M] \subset M$ (that is $[m', m''] \in M, \forall m', m'' \in M$).

c) For a weakly closed subset $M \subset End(V)$ we denote by $\tilde{M} \subset End(V)$ the Lie subalgebra generated by M.

Problem 0.2. *a) If $M \subset End(V)$ is weakly closed, $N \subset M$ and $A \in M$ be such that $[A, \tilde{N}] \subset \tilde{N}$ then $\tilde{N}A \subset A\tilde{N} + \tilde{N}$.*

b) If $N \subset End(V)$ is a nilpotent subset subset then there exists $k > 0$ such that any product of k endomorphisms of V such that $k-1$ of them belong to N is equal to zero.

Given $M \subset End(V), W \subset V$ such that $MW \subset W$ we denote by $M_W \subset End(W), M_{V/W} \subset End(V/W)$ the induced sets of operators.

c) If $M_W \subset End(W), M_{V/W} \subset End(V/W)$ are nilpotent then $M \subset End(V)$ is nilpotent.

Lemma 0.3. *If $M \subset End(V)$ is a weakly closed subset, $N \subset M$ is such that $\tilde{N}$ is nilpotent and $\tilde{N} \neq \tilde{M}$ then there exists $A \in M - M \cap \tilde{N}$ such that $[A, N] \subset \tilde{N}$.*

Proof. Since $\tilde{N} \neq \tilde{M}$ there exists $A_1 \in M - M \cap \tilde{N}$. If $[A_1, N] \subset \tilde{N}$ we can take $A = A_1$. Otherwise there exists $X_1 \in N$ such that $A_2 := [A_1, X_1] \notin \tilde{N}$. Since M is weakly closed we have $A_2 \in M - M \cap \tilde{N}$. If $[A_2, N] \subset \tilde{N}$ we take $A = A_2$ otherwise we repeat the construction. It is clear that either this procedure leads in a finite number of steps to a required element A or we obtain an infinite sequence $A_1...A_n, .. \in M$ such that $A_i = [A_{i-1}, X_{i-1}]$ where $X_{i-1} \in N, A_i \notin \tilde{N}$.

To see that the last possibility does not occur observe that for any $k > 0$ the operator A_k is a linear combination of products of A_1 and $k-1$ factors belonging to N. But it follows from the problem $2b$) that $A_k = 0$ for $k >> 0$. □

Theorem 0.4. *Let k be a field and $M \subset End(V)$ a weakly closed subset such that any element $A \in M$ is nilpotent. Then $\tilde{M} \subset End(V)$ is nilpotent.*

Proof. The proof is by induction in $dim(V)$. The result is true when $dim(V) = 0$ or $M = \emptyset$. So we may assume that $dim(V) > 0$ and

$M \neq \emptyset$. Let Ω be the collection of subsets $N \subset M$ such that $\tilde{N}$ is nilpotent. It is easy to see [please show] that Ω contains a maximal element N [that is $N \subset M$ is such that $\tilde{N}$ is nilpotent but for any $N' \subset M, N' \neq N$ the set $\tilde{N}'$ is not nilpotent] and $\tilde{N} \neq \{0\}$. It is sufficient to show that $\tilde{N} = \tilde{M}$.

Let $W := \tilde{N}V \subset V$. $W \neq \{0\}$ since $\tilde{N} \neq \{0\}$. Moreover $W \neq V$. For otherwise we could write any $v \in$ in the form $v = \sum X_i v_i, X_i \in \tilde{N}, v_i \in V$. Repeating the procedure we will see that for any $k > 0$ we could write any $v \in V$ in the form $v = \sum_i \prod_{j=1}^k X_i^j v_i, X_i^j \in \tilde{N}, v_i \in V$. Since $\tilde{N}$ is nilpotent this would imply that $V = \{0\}$. So $W \neq V$.

Let $R := \{A \in M | AW \subset W\}$. By the construction $N \subset R$.

Problem 0.5. *Show that $\tilde{R}$ is nilpotent.*

A hint. Use the inductive assumptions.

If $N \neq M$ then by Lemma 3 there exists $A \in M - M \cap \tilde{N}$ such that $[A, N] \subset \tilde{N}$. I claim that $A \in R$ [that is $AW \subset W$]. Since $W = \tilde{N}V$ I have to show that $AYv \in W$ for all $Y \in \tilde{N}, v \in V$. By Lemma 2a) we have $AYv = Y_1 Av + Y_2 v, Y_1, Y_2 \in \tilde{N}$. So $AW \subset W$ and $A \in R$.

Since N a maximal element of Ω and [by Problem 5] $\tilde{R}$ is nilpotent we see that $R = N$. So $A \in \tilde{N}$. But by the construction $A \in M - M \cap \tilde{N}$. □

Remark The Engel's theorem follows immediately from the proof Theorem 4[please explain].

Problem 0.6. *Let α be an automorphism of a Lie algebra $\mathfrak{g}$. For any $\lambda \in k$ we define $\mathfrak{g}_\lambda := \{x \in \mathfrak{g} | \alpha(x) = \lambda x\}$.*

a) Show that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda\mu}$.

b)* If $\mathfrak{g}_1 = \{0\}$ and there exists a prime number p such that $\alpha^p = Id_\mathfrak{g}$ then $\mathfrak{g}$ is nilpotent.

c) Construct a Lie algebra $\mathfrak{g}$ and an automorphism α of $\mathfrak{g}$ such that $\mathfrak{g}_1 = \{0\}, \alpha^4 = Id_\mathfrak{g}$ but $\mathfrak{g}$ is not nilpotent.

Problem 0.7. *a) Let $\mathfrak{g}$ be a nilpotent Lie algebra. Show the existence of a chain of ideals $\mathfrak{h}_0 \subset \mathfrak{h}_1 \subset ...\mathfrak{h}_d = \mathfrak{g}$ such that $dim(\mathfrak{h}_i) = i, 0 \leq i \leq d = dim(\mathfrak{g})$.*

b) Let $\mathfrak{g}$ be a solvable Lie algebra and k is algebraically close. Show the existence of a chain of ideals $\mathfrak{h}_0 \subset \mathfrak{h}_1 \subset ...\mathfrak{h}_d = \mathfrak{g}$ such that $dim(\mathfrak{h}_i) = i, 0 \leq i \leq d = dim(\mathfrak{g})$.

c) Give an example of solvable Lie $\mathfrak{g}$ over $\mathbb{R}$ such that there is no chain of ideals $\mathfrak{h}_0 \subset \mathfrak{h}_1 \subset ...\mathfrak{h}_d = \mathfrak{g}$ such that $dim(\mathfrak{h}_i) = i, 0 \leq i \leq d = dim(\mathfrak{g})$.

d) Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field. Show that $\mathfrak{g}$ is not nilpotent iff it contains a two-dimensional non-commutative subalgebra.

I assume that you know the following result.

Lemma 0.8. *Let k be a perfect field, $K \supset k$ an algebraically closed field containing k. Then any element $x \in K$ such that $\gamma(x) = x$ for all $\gamma \in Aut(K/k)$ belongs to k.*

Remark. Any field of characteristic zero is perfect.

Problem 0.9. *Let $K \supset k$ be as in Lemma 8*

a) Let $L \subset K^n$ be a line such that $\gamma(V) = V$ for all $\gamma \in Aut(K/k)$. Show the existence of a line $L \subset k^n$ such that $\tilde{L} = L \otimes_k K$.

b)* Let $\tilde{V} \subset K^n$ be a subspace such that $\gamma(V) = V$ for all $\gamma \in Aut(K/k)$. Show the existence of a k-subspace $V \subset k^n$ such that $\tilde{V} = V \otimes_k K$.

c) Show that for any Lie algebra $\mathfrak{g}$ over a field k and an extension $K \supset k$ there exists unique Lie algebra structure $[,] : \mathfrak{g} \otimes_k K \times \mathfrak{g} \otimes_k K \to \mathfrak{g} \otimes_k K$ such that

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab, x, y \in \mathfrak{g}, a, b \in K$$

We say that the Lie algebra $\mathfrak{g} \otimes_k K$ is obtained from $\mathfrak{g}$ by extension of scalars from k to K.

d) Show that a k-Lie algebra $\mathfrak{g}$ is solvable iff the -Lie algebra $\mathfrak{g} \otimes_k K$ is solvable.

e) Show that $rad(\mathfrak{g} \otimes_k K) = rad(\mathfrak{g}) \otimes_k K$ if k is a perfect field.

A hint. Use the part b) of the problem.

f)* Prove the Cartan criterion for all fields k of characteristic zero of cardinality not greater the the continuum.

Jordan normal form.

For any $\lambda \in k, n \geq 0$ we denote by $A_{\lambda,n}$ the linear operator in the linear space W_n such that V_n has a basis $w_0, ..., w_n$ and $A_{\lambda,n} w_i = \lambda w_i + w_{i+1}$ where $w_{n+1} := 0$.

Theorem 0.10. *Let k be an algebraically closed field, V a finite-dimensional k-vector space and $A \in End(V)$. Then there exists a decomposition $V = \oplus_{i=1}^{r} V_i$ of V into a direct sum of A-invariant subspaces such that for any $i, 1 \leq i \leq r$ the restriction of of A on V_i is equivalent to $A_{\lambda,n}$ for some $\lambda \in k, n \geq 0$.*

Proof. Let $\lambda_1, ..., \lambda_s$ be the set of roots of the characteristic polynomial $P_A(t) := det(A - tId_V)$. For any $i, 1 \leq i \leq s$ we define

$$V^i := \{v \in V | (A - \lambda_i Id_V)^{dim(V)} v = 0$$

Problem 0.11. $V = \oplus_{i=1}^{s} V^i$

We see that it is sufficient to prove the theorem for such operators A that $(A - \lambda_i Id_V)^{dim(V)} = 0$ for some $\lambda \in k$. Therefore we can assume that $A^{dim(V)} = 0$. Let $n \geq 0$ be such that $A^n \neq 0$ but $A^{n+1} = 0$. Choose a vector $w_0 \in V$ such that $A^n w_0 \neq 0$. Let $w_i := A^i w_0$ and $W \subset V$ be the span of of $w_0, ..., w_n$. It is clear that W is an A-invariant subspace of V and the restriction of of A on W is equivalent to $A_{0,n}$. Therefore for the proof of the theorem it is sufficient to show the existence of an A-invariant subspace W' of V such that $V = W \oplus W'$.

To construct W' we fix an linear functional $f : V \to k$ such that $f(w_n) = 1$ and $f(w_i) = 0$ for $i < n$ and consider a linear map $p : V \to W$ given by $p(v) := \sum_{i=0}^{n} f(A^{n-i} v) w_i$.

Problem 0.12. *a)* $p_W = Id_W$.

b) $Ap = pA$.

c) $W' := Ker(p)$ *is* A*-invariant and* $V = W \oplus W'$.

□

Problem 0.13. *Exercises 5.1 and 5.4 from the Kirillov's book.*

The following result is used in the proof of Theorem 5.43 in Kirillov.

Problem 0.14. *Let $\mathfrak{g} \subset V$ be a Lie subalgebra over a field of characteristic zero, $R \subset \mathfrak{g}$ is an ideal and $v \in V - \{0\}$ a vector such that $rv = \lambda(r)v, \lambda(r) \in k, \forall r \in R$. Fix $x \in \mathfrak{g}$ and define $W := span\{v, xv, , x^2v, ...\}$. Then*

a) W *is stable under the action of any* $r \in R$.

b) $\lambda([x, r]) = 0, \forall x \in \mathfrak{g}, r \in R$.

c) $rv^k = \lambda(r)v^k, \forall k \geq 0$ *if* $\lambda(r) = 0$.

Now we can prove Theorem 5.43. I'll consider $\mathfrak{g}/Ker(\rho)$ as a subalgebra in $End(V)$. Let $R = rad(\mathfrak{g})$. Choose $v \in V - \{0\}$ a vector such that $rv = \lambda(r)v, \lambda(r) \in k, \forall r \in R$. It follows from Problem 14 b) that $\lambda([x,r]) = 0$ for any $x \in \mathfrak{g}, r \in R$. Define $\tilde{R} := [R, \mathfrak{g}], W := \{w \in V | \tilde{r}v = 0, \forall \tilde{r} \in \tilde{R}$. Since $\tilde{R} \subset \mathfrak{g}$ is an ideal $\mathfrak{g}W \subset W$. Since V is irreducible and $v \in W$ we see that $W = V$. So $\rho(\tilde{R}) = 0$.