In this lecture we describe finite-dimensional representations of the Lie algebra $sl_2(k)$ in the case when $char(k) = 0$ and k is algebraically closed.

The algebra $sl_2(k)$ has a basis e, f, h such that

$$[e, f] = h, [h, e] = 2e, [h, f] = -2f$$

For any n we denote by V_n the $n + 1$-dimensional space with a basis $v_{n-2k}, 0 \leq k \leq n$ and denote by $\rho_n : sl_2(k) \to End(V_n)$ the linear map such that

$$\rho_n(h)v_{n-2k} = (n - 2k)v_{n-2k}, \rho_n(f)v_{n-2k} = v_{n-2k-2},$$

$$\rho_n(e)v_{n-2k} = k(n - k + 1)v_{n-2k+2}$$

Theorem 0.1. *ρ_n is an irreducible representation of the Lie algebra $sl_2(k)$. Moreover any irreducible representation of $sl_2(k)$ is equivalent to ρ_n for some $n \geq 0$.*

Remark 0.2. *$V_0 = k$ and $\rho_0 \equiv 0$*

Proof. I leave for you to check that ρ_n is a representation of Lie algebra $sl_2(k)$. To prove the irreduciblity of ρ_n we have to show that any invariant non-zero subspace $W \subset V$ is equal to V. Choose a non-zero $\rho_n(h)$-eigenvector $w \in W$. Since all eigenvalues of the operator $\rho_n(h)$ are distinct we have $w = cv_{\lambda-2k}$ for some $k, 0 \leq k \leq n$ and some non-zero $c \in k$. But then vectors $\rho_n(f)^a w, \rho_n(e)^b w \in W$ span V.

Let $\rho : sl_2(k) \to End(V)$ be an irreducible finite-dimensional representation of $sl_2(k)$. We write $E = \rho(e), F = \rho(f), H = \rho(h)$. For any $\lambda \in k$ we define

$$V[\lambda] := \{v \in V | Hv = \lambda\}$$

Lemma 0.3. $EV[\lambda] \subset V[\lambda + 2], FV[\lambda] \subset V[\lambda - 2]$

Proof. If $v \in V[\lambda]$ we have $HEv = [H, E]v + EHv = 2Ev + E\lambda v = (\lambda + 2)EV$. Analogous arguments show that $HFv = (\lambda - 2)FV$. $\square$

Since V is finite-dimensional and $char(k) = 0$ there exists $\lambda \in k$ such that $V[\lambda] \neq \{0\}$ and $V[\lambda + 2] = \{0\}$. Choose a non-zero vector $v_\lambda \in V[\lambda]$. Since $V[\lambda + 2] = \{0\}$ we have $Ev_\lambda = \{0\}$. For any $k \geq 0$ define $v_{\lambda-2k} := F^k(v_\lambda)$.

Lemma 0.4.

$$Hv_{\lambda-2k} = (\lambda - 2)k, Fv_{\lambda-2k} = v_{\lambda-2k-2}, Ev_{\lambda-2k} = k(\lambda - k + 1)v_{\lambda-2k+2}$$

Proof. The first equality follows from Lemma 2, the second from the definition of vectors $v_{\lambda-2k}$. We prove the equality

$(\star_k)$ $Ev_{\lambda-2k} = k(\lambda-k+1)v_{\lambda-2k+2}$

by induction in k. By the definition of v_λ the equality $(\star_0)$ is true. Assume that $Ev_{\lambda-2k+2} = (\lambda-k)(k-1)v_{\lambda-2k+4}$ the equality $(\star_{k-1})$ is true. Then we have

$$Ev_{\lambda-2k} = EFv_{\lambda-2k+2} = [E,F]v_{\lambda-2k+2} + FEv_{\lambda-2k+2} =$$
$$Hv_{\lambda-2k+2} + F(\lambda-k)(k-1)v_{\lambda-2k+4} =$$
$$(\lambda-2k+2)v_{\lambda-2k+2} + (\lambda-k)(k-1)Fv_{\lambda-2k+4} =$$
$$k[(\lambda-2k+2)+(\lambda-k)(k-1)]v_{\lambda-2k+2} = k(\lambda-k+1)v_{\lambda-2k+2}$$

□

Since V is finite-dimensional and [by Lemma 2] $F^k v_\lambda \in V[\lambda-2k]$ there exists $n \geq 0$ such that $F^n v_\lambda \neq \{0\}$ but $F^{n+1}v_\lambda = \{0\}$.

I claim that $\lambda \in \mathbb{Z}_{\geq 0}$. Really if $\lambda \notin \mathbb{Z}_{\geq 0}$ then $(\lambda-k+1)k \neq 0$ for all $k > 0$. From this we deduce by induction in k that $v_{n-2k} \neq 0, \forall k \geq 0$. But is impossible since $dimV < \infty$.

So we can assume that $\lambda = n \in \mathbb{Z}_{\geq 0}$. I claim that the set

$$\{v_{n-2k}\}, 0 \leq k \leq n, v_{n-2k} := F^k(v_\lambda)$$

is a basis of V. Really it follows from Lemma 3 that vectors $v_{n-2k}, 0 \leq k \leq n$ are linearly independent and that they generate an $sl_2(k)$-invariant subspace of V. Since V is irreducible vectors $v_{n-2k}, 0 \leq k \leq n$ span V. It is clear now that the representation ρ is equivalent to the representation ρ_n.

□

Definition 0.5. *We define the Casimir element* $\Delta \in U(sl_2(k))$ *by* $\Delta := ef + fe + h^2/2 \in U(sl_2(k))$

Problem 0.6. *Show that*

a) The representations ρ_n *are equivalent to ones constructed in the end of the lecture 1.*

b) Δ *belongs to the center of the algebra* $U(sl_2(k))$.

c) $\rho_n(\Delta) = \frac{n(n+2)}{2} Id_{V_n}$

Lemma 0.7. *Every exact sequence*

$$\{0\} \to V' \to V \to k \to \{0\}$$

of finite-dimensional representations of $sl_2(k)$ *splits [that is* V *is equivalent to the direct sum* $V' \oplus k$*].*

Proof. a) Consider first the case when the representation V' is irreducible. As follows from Theorem 1 it is sufficient to show that any exact sequence of representations of $sl_2(k)$ of the form

$$\{0\} \to V_n \to V \to k \to \{0\}$$

splits. We consider separately the cases $n = 0$ and $n \neq 0$.

a') Assume that $n = 0$. So we have an exact sequence

$$\{0\} \to V_0 \to V \to k \to \{0\}$$

[I write V_0 instead of k to distinguish between the subspace (V_0) and the quotient space (k) of V.] Choose a a basis v_1, v_2 of V such that $v_1 \in V_0$. Then we can write elements of $End(V)$ as 2×2 matrices. Let ρ be the representation of $sl_2(k)$ on V. Since $\rho_0 \equiv 0$ we obtain

$$\rho(x) = \begin{pmatrix} 0 & a(x) \\ 0 & 0 \end{pmatrix}, x \in sl_2(k)$$

So $\rho([x, y]) = [\rho(x), \rho(y)] = 0, \forall x, y \in sl_2(k)$ and therefore $\rho(e) = \rho(f) = \rho(h) = 0$. So $r \equiv 0$ and any linear section $s : k \to V$ defines a splitting $V = V_0 \oplus k$ representations of $sl_2(k)$.

a") If $n \neq 0$ consider the operator $A := \rho(\Delta) \in End(V)$. It is clear that the subspace $V_n \subset V$ is A-invariant and it follows from Problem 5 c) that the restriction of A on V_n is equal to $\frac{n(n+2)}{2} Id_{V_n}$. On the other hand A acts on the quotient space $V/V_n = k$ as $\rho_0(\Delta) = 0$. Since $\frac{n(n+2)}{2} \neq 0$ the space $W := \{v \in V | Av = 0\} \subset V$ is a non-zero and $V = V_n \oplus W$. As follows from Problem 5 b) the subspace W is $sl_2(k)$-invariant. So $V = V_n \oplus k$.

Now we prove Lemma 7 by induction in $dim(V)$. Consider an exact sequence $\{0\} \to V' \to V \to k \to \{0\}$ of representations of $sl_2(k)$. We know that it splits if the representation of $sl_2(k)$ on V' is irreducible. If V' is reducible choose an irreducible subrepresentation $W \subset V'$ and consider the exact sequence $\{0\} \to V'/W \to V/W \to k \to \{0\}$. By the inductive assumptions it splits and there exist an $sl_2(k)$-equivariant section $s : k \hookrightarrow V/W$. Let $\tilde{V} \subset V$ be the preimage of $Im(s)$. We have an exact sequence $\{0\} \to V' \to \tilde{V} \to k \to \{0\}$. Using once more the inductive assumption we see that there exists an $sl_2(k)$-equivariant section $\tilde{s} : k \hookrightarrow V$.

□

Theorem 0.8. *Any finite-dimensional representation of the Lie algebra $sl_2(k)$ is completely reducible.*

Proof. We have to show that for any exact sequence

$$\{0\} \to V' \to V \to V'' \to \{0\}$$

of representations of $sl_2(k)$ there exists an $sl_2(k)$-equivariant section $\tilde{s} : V'' \to V$ of the projection $p : V \to V''$.

Consider the exact sequence

$$\{0\} \to V' \otimes V''^{\vee} \to V \otimes V''^{\vee} \to V'' \otimes V''^{\vee} \to \{0\}$$

Let $kId_{V''} \subset V'' \otimes V''^{\vee} = End(V'')$ be the subspace of scalar operators and $\{0\} \to V' \otimes V''^{\vee} \to W \to kId_{V''} \to \{0\}$ be the exact sequence where $W := q^{-1}(kId_{V''}) \subset V \otimes V''^{\vee}$ and $q : V \otimes V''^{\vee} \to V'' \otimes V''^{\vee}$ is the map induced by the projection $p : V \to V'' = V/V'$. As follows from Lemma 7 there exists an $sl_2(k)$-equivariant section $s : kId_{V''} \hookrightarrow V \otimes V''^{\vee}$ of the projection q. Let $s' := s \otimes Id_{V''} : V'' \hookrightarrow V \otimes V''^{\vee} \otimes V''$ and

$$\tilde{s} := (Id_V \otimes Tr_{V''}) \circ s' : V'' \to V$$

where

$$Tr_{V''} : V''^{\vee} \otimes V' = End(V'') \to k$$

is the trace map. I'll leave for you to prove that $\tilde{s} : V'' \to V$ is an $sl_2(k)$-equivariant section of the projection $p : V \to V''$. □

Problem 0.9. *a) Complete the proof of Theorem 8.*

b) Extend the proof of Theorems 1 and 8 to the case when k is an arbitrary field of characteristic zero.

c) Let $\rho : sl_2(k) \to End(V)$ be a finite-dimensional representation. Show that the operators $\rho(e), \rho(f) \in End(V)$ are nilpotent and there exists unique representation $\tilde{\rho} : SK(2, k) \to Aut(V)$ such that

$$\tilde{\rho}(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}) = \exp(a\rho(e))$$

and

$$\tilde{\rho}(\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}) = \exp(b\rho(f))$$

From now on we assume that k is an arbitrary closed field of arbitrary characteristic.

d) Let $\mathfrak{g}$ be a Lie algebra, $\rho : \mathfrak{g} \to End(V)$ a finite-dimensional irreducible representation and $A \in End_{\mathfrak{g}}(V)$ [that is A is a linear operator such that $A\rho(x) = \rho(x)A, \forall x \in \mathfrak{g}$]. Show that $A = cId_V$ for some $c \in k$.

e) Give an example of a reducible 2-dimensional representation ρ of a Lie algebra $\mathfrak{g}$ such that $End_{\mathfrak{g}}(V) = kId_V$.

f) Let $\mathcal{Z}U(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Show that for any irreducible representation $\rho : \mathfrak{g} \to End(V)$ and any $z \in \mathcal{Z}U(\mathfrak{g})$ we have $\rho(u) = c_\rho(z)Id, c_\rho(z) \in k$.

g) Assume that $char(k) = p >$. Show that $e^p, f^p, h^p - h \in \mathcal{Z}sl_2(k)$.

h)* Show that any irreducible representations of $sl_2(k)$ such that $\rho(e^p) = \rho(f^p) = \rho(h^p - h) = 0$ is equivalent to a representation ρ_n from the problem 12 in the first lecture for unique $n < p$.

i) Show that any exact sequences $\{0\} \to V_i \to V \to V_j \to \{0\}$ of representations of $sl_2(k)$ splits if $i + j \neq p - 2$.

j)* Construct an example of a non-split exact sequences $\{0\} \to V_i \to V \to V_j \to \{0\}$ for $i + j = p - 2$

k) The Lie algebra $sl_2(k)$ is solvable if $char(k) = 2$. Where does the proof of theorem of Lie fail?

l) Prove theorem 5.33 from the book of A.Kirillov.

From now on [at least for a while] we will follow the book of A.Kirillov available on www.math.sunysb.edu/ kirillov/liegroups/. We start from the chapter 5.3 [page 75]. A couple of remarks. When written

a) *complex* read *an algebraically closed field $\bar{k}$ of characterisitc zero.*

b) *real* read *a field k of characterisitc zero* such that $\bar{k}$ is the closure of k.

c) *complexification* of a k-Lie algebra $\mathfrak{g}$ read the $\bar{k}$-Lie algebra $\mathfrak{g}_k \otimes \bar{k}$.

An explanation [the proof of Proposition 5.31]. One should say

We claim that W is stable under the action of any $h \in \mathfrak{g}'$;

(5.10) $hv^k = \lambda(h)v^k + \sum_{l<k} a_{kl}(h)v^l$ and moreover if $\lambda(ad_x^j(h) = 0, \forall j > 0$ then $hv^k = \lambda(h)v^k, \forall k \geq 0$

The proof is by induction in k.