Problem 0.1. *a) Let $R \subset E$ be a set of vectors satisfying the conditions $(R1), (R2)$ and $(R3)$ and $\alpha, c\alpha \in R, c \in \mathbb{R}$. Show that the only possible values of c are $\pm 1/2, \pm 1, \pm 2$.*

b) Let $R \subset E$ be a root system and $\{\alpha_i\}, i \in I$ be the set of simple roots corresponding to a polarization $R = R^+ \cup R^-$. We define

$$R^\vee := \{v \in E^\vee | < v, \beta > \in \mathbb{Z}\}, \forall \beta \in R$$

Show that

$\beta^\vee \in R^\vee$ for all $\beta \in R$

$R^\vee$ is a root system

$\{\alpha_i^\vee\}, i \in I$ is a set of simple roots for $R^\vee$.

c) Show that map $W \to \pm 1, w \to -1^{l(w)}$ is a group homomorphism.

d) Let $v = \sum_i k_i \alpha_i, k_i \in \mathbb{Z}_{\geq 0}$ be a non-negative combination of simple roots which is not a multiple of a root. Show the existence of $w \in W$ such that $w(v) = \sum_i k_i' \alpha_i$ where some of $\{k_i'\}$ are positive and some are negative.

A hint. Let $L = \{x \in E | (v, x) = 0\}$. Since v is not a multiple of a root we can find a regular point $v' \in L$. Then the element $w \in W$ be such that $w(v) \in C_+$ is the solution of d).

We say that $g \in Aut(E)$ is a reflection if $g^2 = e$ and $dim(E^g) = dim(E) - 1$ where $E^g := \{x \in E | g(x) = x\}$.

e) Show that any reflection $w \in W$ has a form $w = s_\beta, \beta \in R$.

f) Let $E = \mathbb{R}^8, e_1, \dots, e_8$ the standard basis of E

$$\Lambda' := \mathbb{Z}(e_1 + \dots + e_8) + = \mathbb{Z}^8 \Lambda \subset E$$

and $\Lambda \subset \Lambda'$ consists of elements

$$\sum_{i=1}^{8} c_i e_i + c(e_1 + \dots + e_8)$$

such that $\sum_{i=1}^8 c_i$ is even. Let $R := \{v \in \Lambda | (v, v) = 2\}$.

Show that $\Lambda \subset \Lambda'$ is a subgroup.

$$R = \{\pm e_i \pm e_j\}, i \neq j \cup \{1/2 \sum_{i=1}^{8} \epsilon_i e_i\}$$

where $\epsilon_i = \pm 1$ and $\prod_{i=1}^8 \epsilon_i = 1$.

R is a root system and the set

$$\{1/2[e_1-(e_2+\cdots+e_7)+e_8], e_1+e_2, e_{j+1}-e_j\}, 1\le j\le 6$$

is the set of simple roots of the type E_8.

f) Construct root systems of the types E_6, E_7.

Let $R\subset E$ be a root system, $\Delta=\{\alpha_i\}, i\in I$ is a set of simple roots and $\sigma\in AutI$ such that $(\alpha_i,\alpha_j)=(\alpha_{\sigma(i)},\alpha_{\sigma(j)}),\forall i,j\in I$ and $\Gamma\subset of Aut(\Delta)$ be the group generated by σ. We also denote by $\sigma\in Aut(E)$ the linear map such that $\sigma((\alpha_i)=\alpha_{\sigma(i)}, i\in I$. Let $\tilde{I}=I/\{\sigma\}^{\mathbb{Z}}$ be the set of orbits of the action of σ on I. We assume that for any $i\in I, k\in\mathbb{Z}$ such that $\sigma^k(i)\neq i$ the roots $\alpha_i,\alpha_{\sigma^k(i)}$
g) Show that set

$$\{\tilde{\alpha}_{\tilde{i}}\}:=\sum_{i\in\Omega_{\tilde{i}}}\alpha_i\in E^\sigma, \tilde{i}\in\tilde{I}$$

is a set of simple roots for a root system $\tilde{R}$ in E^σ. Moreover the vertices of the Dynkin diagram $\tilde{I}$ of $\tilde{R}$ are Γ-orbits in I and $(\tilde{\alpha}_{\tilde{i}},\tilde{\alpha}_{\tilde{i}})=|\in\Omega_{\tilde{i}}$.

h) Construct the root system F_4 from the root system E_6 and the root system G_2 from the root system D_4

Remark. Let (E,R) be the root system of the type E_8 and $\Lambda\subset E$ be the lattice [the span of a basis] generated by simple roots. Then

$Q(\lambda,\lambda)\in 2\mathbb{Z},\forall\lambda\in\Lambda$ where $\Lambda\subset E$ and $det(A)=1$ where $Q(v,v'):=(v,v')$

One can show that such pairs $(Q,\Lambda), Q:E\times E\to\mathbb{R},\Lambda\subset E$ exists only if $dim(E)\equiv 0(mod8)$. In the case when $dim(E)=8$ any such pair is isomorphic to the one coming from E_8, in in the case when $dim(E)=16$ any such pair is isomorphic either to the one coming from $E_8\oplus E_8$ or to the one coming from D_{16}, in the case when $dim(E)=24$ there are 24 classes of such pairs and in the case when $dim(E)=32$ there are more then 10^7 such pairs.

Definition 0.2. *Let V be a [not necessarily finite-dimensional] vector space over a field k of characteristic zero. We say that a linear operator $T:V\to V$ is locally nilpotent if for any $v\in V$ there exists $N>0$ such that $A^Nv=0$*

Remark. If $T:V\to V$ is locally nilpotent then we can define $\exp(T):=\sum_{n>\geq 0}T^n/n!$.

Problem 0.3. *Let $\mathfrak{g}$ be a Lie algebra with generators $x_i, i \in I$ and $D : \mathfrak{g} \to \mathfrak{g}$ a differentiation.*

a) If there exists $N > 0$ such that $D^N x_i = 0, \forall i \in I$ then D is locally nilpotent.

b) If D is locally nilpotent then $\exp(D)$ *is an automorphism of* $\mathfrak{g}$.

c) Let $\rho : sl_2(k) \to End(V)$ be a representation, $v \in V$ be such that $\hat{e}v = 0, \hat{h}v = nv, n \geq 0$. Then $\hat{e}\hat{f}^{n+1}v = 0$. If $dim(V) < \infty$ then $\hat{f}^{n+1}v = 0$.

d) Assume that the representation $\rho : sl_2(k) \to End(V)$ is such that $\hat{e}, \hat{f}$ are locally nilpotent. Then for any vector $v \in V$ there exists a finite-dimensional $sl_2(k)$-invariant subspace $V' \subset V$ containing v.

e) Let $\rho : sl_2(k) \to End(V)$ be as in d), $s := \exp(e)\exp(-f)\exp(e) \in Aut(V), v \in V$ by such that $\hat{h}v = nv, n \in \mathbb{Z}$. Then $\hat{h}sv = -nsv$.

a)

Problem 0.4. *a) Let V, W be k- vector spaces with filtrations*

$$V_0 \subset \cdots \subset V_n \subset \cdots \subset V, W_0 \subset \cdots \subset_n \subset W_n \cdots \subset W$$

such that $V = \cup_n V_n, W = \cup_n W_n$. Let $T : V \to W$ be a linear map such that $T(V_n) \subset W_n$ and such that the induced maps $T_n : V_n/V_{n-1} \to W_n/W_{n-1}$ are bijections. Then $T : V \to W$ is a bijection.

b) Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}', \mathfrak{g}'' \subset \mathfrak{g}$ Lie subalgebras such that $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$ as a vector space. Then the natural [product] map $U(\mathfrak{g}') \otimes U(\mathfrak{g}'') \to U(\mathfrak{g})$ a bijection.