We start with some definitions and problems from linear algebra. For simplicity I assume that $char(k) \neq 2$. Please notice that a number of definitions are in the middle of homework assignments.

Definition 0.1. *Let V, W be k-vector spaces.*

a) We denote by $V^\vee := Hom(V, k)$ the dual vector space of linear functionals $\lambda : V \to k$.

b) For any linear map $T : V \to W$ we denote by $T^\vee : W^\vee \to V^\vee$ the map given by $T^\vee(\lambda)(v) := \lambda(T(v)), \lambda \in W^\vee, v \in V$. We say that $T^\vee$ is the map dual to T.

c) Let $B : V \times V \to k$ be a bilinear form. We say that Q is non degenerate if for any $v' \in V - \{0\}$ there exists $v'' \in V$ such that $B(v', v'') \neq 0$.

d) Let $Q : V \to k$ be a quadratic form. We associate with Q a symmetric bilinear form

$$B_Q : V \times V \to k, B_Q(v, w) := Q(v + w) - Q(v) - Q(w)$$

We say that the quadratic form Q is non-degenerate if the bilinear form B_Q is non-degenerate.

e) We say that a skew-symmetric bilinear form $B : V \times V \to k$ is symplectic if it is non-degenerate.

Problem 0.2. *a) Construct a natural linear map $V \to (V^\vee)^\vee$ and show that this map is an isomorphism if V is finite-dimensional.*

b) If U, V, W are k-vector spaces and $S : U \to V, V \to W$ are linear maps then $(T \circ S)^\vee = S^\vee \circ T^\vee$.

c) Let V be a finite-dimensional vector space, $B : V \times V \to k$ a non-degenerate bilinear form. Show that for any $v'' \in V - \{0\}$ there exists $v' \in V$ such that $B(v', v'') \neq 0$.

$d^\star$) Is the assumption of the finite-dimensionality is important for the validity of c).

Let k be a field, V', V'' be k-vector spaces. For any k-vector space W we denote by $B(V', V''; W)$ the k-vector space of bilinear forms $b : V' \times V'' \to W$.

Definition 0.3. *A tensor product of V' and V'' is a pair (V, m) where V is a k-vector space and $m : V' \times V'' \to V$ a bilinear form such that for any k-vector space W and a bilinear form $b : V' \times V'' \to W$ there exists unique linear map $T : V \to W$ such that $b(v', v'') \equiv T(m(v', v''))$.*

Problem 0.4. *Show that*

a) In the case when a tensor product (V, m) of vector spaces V' and V'' exists it is well defined

[you have to show that if $(\tilde{V}, \tilde{m})$ is an another tensor product (V, m) of V' and V'' then there exists unique linear map $S : V \to \tilde{V}$ such that

$$S((m(v', v'')) \equiv \tilde{m}(v', v'')$$

and moreover the map S is an isomorphism].

Since the tensor product (V, m) of V' and V'' is well defined we can talk about the tensor product of vector spaces V' and V'' which we denote by $V' \otimes V''$ [we did not yet show that the tensor product $V' \otimes V''$ exists] and write $v' \otimes v'' \in V' \otimes V''$ instead of $m(v', v'')$.

b) for any k-vector spaces V' and V'' the tensor product $V' \otimes V''$ exists

[A hint: Use the existence of bases e'_i, e''_j for vector spaces V', V''.]

c) If V', V'' are finite-dimensional k-vector spaces then

$$dim(V' \otimes V'') = dim(V')dim(V'')$$

d^*) Let A be a commutative ring and M', M'' be A-modules. Give a definition of the tensor product $M' \otimes_A M''$ and prove the uniqueness and the existence of the tensor product $M' \otimes_A M''$.

Definition 0.5. *a) Let k be a field. A k-Lie algebra is a pair $(\mathfrak{g}, [,])$ where $\mathfrak{g}$ is a k-vector space and $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear map such that*
$[x, x] = 0, x \in \mathfrak{g}$ *(that is [,] is skew-symmetric) and*
$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0, x, y, z \in \mathfrak{g}$ *(the Jacobi identity).*

We often say "a Lie algebra" instead of "a k-Lie algebra" and write $\mathfrak{g}$ instead of $(\mathfrak{g}, [,])$.

b) We define $\mathfrak{z}(\mathfrak{g}) := \{x \in \mathfrak{g} | [x, y] = 0, \forall y \in \mathfrak{g}$. We say that $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$.

c) Let $(\mathfrak{g}, [,])$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be a subspace. We say that $\mathfrak{h}$ is a Lie subalgebra if $[h', h''] \in \mathfrak{h}, \forall h', h'' \in \mathfrak{h}$ and say that $\mathfrak{h}$ is an ideal if $[x, h] \in \mathfrak{h}, \forall x \in \mathfrak{g}, h \in \mathfrak{h}$.

d) Let $\mathfrak{h}, \mathfrak{g}$ be Lie algebras. A linear map $f : \mathfrak{h} \to \mathfrak{g}$ is called a Lie algebra homomorphism (or simply a homomorphism) if

$$[f(x), f(y)] = f([x, y]), x, y \in \mathfrak{h}.$$

e) We say that a homomorphism f is an isomorphism if f is one-to-one and onto.

Problem 0.6. *a) Show that for any Lie algebra homomorphism $f : \mathfrak{h} \to \mathfrak{g}$ the image $Im(f) \subset \mathfrak{g}$ of a homomorphism f is a Lie subalgebra and the kernel $Ker(f) \subset \mathfrak{h}$ is an ideal.*

b) Let A be an associative k-algebra. We define the map $[,] : A \times A \to A$ by $[a, b] := ab - ba, a, b \in A$. Show that $(A, [,])$ is a Lie algebra.

Definition 0.7. *a) In the case when $A = End_k(V)$ is the algebra of endomorphisms of a k- vector space V we denote the Lie algebra $(End(V), [,])$ by $gl(V)$.*

b) In the case when V is finite-dimensional we denote by $sl_n(V) \subset gl(V)$ the subspace of endomorphisms $T \in End(V)$ such that $Tr(T) = 0$.

c) If $V = k^n$ we write $gl_n(k)$ instead of $gl(k^n)$.

d) If $Q : V \to k$ is a non-degenerate quadratic form we denote by $so_Q \subset gl(V)$ of endomorphisms $T \in End(V)$ such that $B_Q(Tv', v'') + B_Q(v', Tv'') = 0, \forall v', v'' \in V$.

e) If $B : V \times V \to k$ is a symplectic form we denote by $sp_B \subset gl(V)$ of endomorphisms $T \in End(V)$ such that $B(Tv', v'') + B(v', Tv'') = 0, \forall v', v'' \in V$.

Problem 0.8. *Show that y*

a) $sl_n(k)$ is an ideal of $gl_n(k)$.

b) Let $\mathfrak{g}$ be the 3-dimensional k-vector space with the basis (f, e, h) and $[,]$ be the bilinear skew-symmetric map such that

$$[e, f] = h, [h, e] = 2e, [h, f] = -2f$$

Prove that $(\mathfrak{g}, [,])$ is a Lie algebra and that it is isomorphic to the Lie algebra $sl_2(k)$.

c) Let $so_n := so_{Q_n}$ where Q_n is the quadratic form on $V = k^n$ given by $Q(x_1, ..., x_n) := \sum_{i=1}^n x_i^2$. Show that so_n is the set of skew-symmetric matrices.

d) Let $sp_{2n} := sp_{B_n}$ where B_n is the symplectic form on $V = k^{2n}$ given by

$$B_n(x_1, ..., x_{2n}; y_1, ..., y_{2n}) := \sum_{i=1}^n (x_i y_{i+n} - x_{i+n} y_i)$$

Describe the subset $sp_B \subset gl_{2n}(k)$.

e) Show that for $so_Q \subset gl(V)$ *is a Lie subalgebra for any quadratic form* Q.

f) Show that for $sp_B \subset gl(V)$ *is a Lie subalgebra for any symplectic form* B.

g) Find all the ideals and subalgebras of the Lie algebra $sl_2(k)$

h) Classify all the Lie algebras of dimension* ≤ 3 *over an algebraicly closed field* k.

[that is construct a set of k*-Lie algebras* $\mathfrak{g}_i, 1 \leq i \leq N$ *such that the Lie algebras* $\mathfrak{g}_i, \mathfrak{g}_j$ *are not isomorphic for* $i \neq j$ *and any* k*-Lie algebra of dimension* ≤ 3 *is isomorphic to* $\mathfrak{g}_i$ *for some* $i, 1 \leq i \leq N$*].*

i) Show that Lie algebras so_3 *and* sl_2 *are isomorphic iff [if and only if] there exists* $i, j \in k$ *such that* $i^2 + j^2 = -1$.

Definition 0.9. *Let* $\mathfrak{g}$ *be a* k*-Lie algebra. We denote by* $\mathcal{D}(\mathfrak{g})$ *the set of* k*-linear maps* $D : \mathfrak{g} \to \mathfrak{g}$ *such that* $D([x, y]) = [x, Dy] + [Dx, y]$ *for all* $x, y \in \mathfrak{g}$*. We call elements of the set* $\mathcal{D}(\mathfrak{g})$ *differentiations of* $\mathfrak{g}$*.*

Problem 0.10. *Show that*

a) For any $D', D'' \in \mathcal{D}(\mathfrak{g})$ *the map*

$$[D', D''] : \mathfrak{g} \to \mathfrak{g}, [D', D''](x) := D' \circ D''(x) - D'' \circ D'(x)$$

belongs to $\mathcal{D}(\mathfrak{g})$.

b) The map $(D', D'') \to [D', D'']$ *defines a Lie algebra structure on the vector space* $\mathcal{D}(\mathfrak{g})$*. We call it the Lie algebra of differentiations of* $\mathfrak{g}$ *and denote by* $Diff(\mathfrak{g})$.

c) For any $x \in \mathfrak{g}$ *the map*

$$ad_{\mathfrak{g}}(x) : \mathfrak{g} \to \mathfrak{g}, y \to [x, y]$$

belongs to $Diff(\mathfrak{g})$ *and the map* $ad_{\mathfrak{g}} : \mathfrak{h} \to Diff(\mathfrak{g})$ *is a Lie algebra homomorphism.*

d) The map $ad_{sl_2(k)}$ *is an isomorphism iff [=if and only if]* $char(k) \neq 2$

Definition 0.11. *Let* $\mathfrak{g}$ *be a* k*-Lie algebra.*

a) A representation of $\mathfrak{g}$ *on a* k*- vector space* V *is a Lie algebra homomorphism* $\rho : \mathfrak{g} \to End(V)$.

b) A subspace $W \subset V$ *is* $\rho - invariant$ *if* $\rho(x)w \subset W$ *for all* $x \in \mathfrak{g}, w \in W$.

We often say "invariant" instead of "ρ -invariant".

c) A representation $\rho : \mathfrak{g} \to End(V)$ is irreducible if for any ρ - invariant subspace $W \subset V$ either $W = V$ or $W = \{0\}$

Let V_n be the space of homogeneous polynomial $P(u, v)$ of degree n over the filed k. Consider a k-linear map $\rho_n : sl_2(k) \to End(V_n)$ given by

$$\rho_n(f)(P) = u\partial/\partial v(P),$$
$$\rho_n(e)(P) = v\partial/\partial u(P)$$

,

$$\rho_n(h)(P) = -u\partial/\partial u(P) + v\partial/\partial v(P)$$

Problem 0.12. *Show that*

a) the map ρ_n is a representation of the Lie algebra $sl_2(k)$.

b) the representation ρ_n is irreducible if $char(k) = 0$.

$c^\star$) If $p := char(k) \neq 0$ then the representation ρ_n is irreducible if and only if $p > n$.