Let $R \subset E$ be a reduced root system, $R = R^+ \cup R^-$ a polarization, $\Sigma = \{\alpha_i\}, i \in I$ the simple roots, $C^+ \subset E$ the positive Weyl chamber, W the Weyl group. You know that W is generated by simple reflections $s_i, i \in I$ which satisfy the relations [see the problem 7.11]

$$s_i^2 = e, (s_i s_j)^{m_{ij}} = e, i, j \in I$$

Let $\tilde{W}$ be the group generated by elements $\tilde{s}_i, i \in I$ and relations

$$\tilde{s}_i^2 = e, (\tilde{s}_i \tilde{s}_j)^{m_{ij}} = e, i, j \in I$$

It is clear that there exists unique group homomorphism $p : \tilde{W} \to W$ such that $p(\tilde{s}_i) = s_i, i \in I$.

Theorem 0.1. *The map $p : \tilde{W} \to W$ is an isomorphism.*

Proof. Since the group W is generated by $s_i, i \in I$ the homomorphism p is surjective. So we have to show that $Ker(p) = \{e\}$. The proof will use the following result from Topology.

Lemma 0.2. *Let $X \subset \mathbb{R}^d$ be a finite union of linear [or affine] subspaces of dimension $< d - 2$. Then $\pi_1(\mathbb{R}^d - X) = \{e\}$.*

Let $Z := \cup_{\beta \in R^+} L_\beta, E^0 := E - Z$ be the set of regular elements of E. For any pair $\beta \neq \beta' \in R^+$ we define $Y_{\beta\beta'} := L_\beta \cap L_{\beta'}$, write $Y := \cup_{\beta \neq \beta'} Y_{\beta\beta'}$ and define $X := \cup_{\beta,\beta',\beta''} L_\beta \cap L_{\beta'} \cap L_{\beta''}$ where $\beta, \beta', \beta'' \in R^+$ run through all distinct triples. We write $Y^0_{\beta\beta'} = Y_{\beta\beta'} - (Y_{\beta\beta'} \cap X)$. It is clear that $Y - X$ is a disjoint union of $Y^0_{\beta\beta'}$.

Let $\gamma : [0, 1] \to E - Y$ be a continuous map [a curve] such that $f(0) \in C_+$ and $f(1) \in E^0$. Then we can define $\tilde{w}_\gamma \in \tilde{W}$ as follows. Let $0 < a_1 < \cdots < a_r < 1$ be points such that $f(a_j) \in Z$. Since $f(a_j) \in Z - Y$ there exists unique $\beta_j \in R^+$ such that $f(a_j) \in L_{\beta_j}$. As in Lemma 7.31 we obtain a sequence of simple roots $\alpha_{i_1}, \dots, \alpha_{i_r}$ such that

$$\beta_j = s_{i_1} \dots s_{i_{j-1}}(\alpha_{i_j}), 1 \leq j \leq r$$

We define $\tilde{w}_\gamma := \tilde{s}_{i_1} \dots \tilde{s}_{i_r} \in \tilde{W}$.

Conversely given an elements $\tilde{w} \in \tilde{W}$ and a represtnation $[\tilde{w}]$ of $\tilde{w}$ as a product $\tilde{w} = \tilde{s}_{i_l} \dots \tilde{s}_{i_1} \in \tilde{W}$ we define a curve $\gamma([\tilde{w}]) \subset E - Y$ as the union of intervals connecting $w_r(t)$ with $w_{r+1}t, 1 \leq r \leq r$ where $t \in C^+$ is a regular element. [Please check that $\gamma([\tilde{w}]) \subset E - Y$]. It is easy to see that $\tilde{w}_{\gamma([\tilde{w}])} = \tilde{w}$.

Example.1) Assume that $dim(E) = 2$ and γ is a simple loop around 0 such that $f(0) \in C_+$. Then $\tilde{w}_\gamma := (s_i s_j)^{m_{ij}} = e$.

2) For an arbitrary root system choose a pair of distinct simple root α_i, α_j and consider a simple loop γ in $E - Y$ around $Y^0_{\alpha_i\alpha_j}$ with the beginning in C^+. Then $\tilde{w}_\gamma = e$.

Claim. If γ is a loop $[f(0) = f(1)]$ then $\tilde{w}_\gamma = e$.

The Claim implies the Theorem.

Proof. Let $\tilde{w} \in \tilde{W}$ be such that $p(\tilde{w}) = e$. Choose a represtnation $[\tilde{w}]$ of $\tilde{w}$ as a product $\tilde{w} = \tilde{s}_{i_l} \dots \tilde{s}_{i_1} \in \tilde{W}$ and consider the curve $\gamma([\tilde{w}])$. Since $p(\tilde{w}) = e$ it is a loop. Therefore $\tilde{w}_{\gamma([\tilde{w}])} = e$. But $\tilde{w}_{\gamma([\tilde{w}])} = \tilde{w}$. □

Proof of the Claim. Let S^1 be the circle obtained from the interval $[0, 1]$ by gluing together 0 and 1. We can consider γ as a continuous map $f : S^1 \to E - Y$. By Lemma 2, $\pi_1(E - X) = \{e\}$ and therefore the loop γ is contractible in $E - X$. So there exists a continuous family $f_a : S^1 \to E - X$ of loops such that $f_0 \equiv f, f_a(0) = f_a(1) \equiv t$ and $f_1 \equiv t$. We can assume that there is a finite set $A \subset (0, 1), A = \{a_1 < \dots a_N\}$ such that $Im(f_a) \subset E - Y$ for $a \in [0, 1] - A$ and for any $q, 1 \leq q \leq N$ and $f_a(x) \in E - Y, \forall a \in [0, 1], x \neq 1/2$. For any $q, 1 \leq q \leq N$ we write $v_q := f_{a_q}(1/2) \in Y$. [Students who took topology please check the validity of this claim].

Since $v_q \in Y - X$ there exist unique pair $\beta_q \neq \beta'_q \in R^+$ such that $v_q \in Y_{\beta_q\beta'_q}$. For any $q, 1 \leq q < N$ we choose $b_q \in (a_q, a_{q+1})$ and define $b_0 = 0, b_{N+1} = 1$. We define $\gamma_q := f_{b_q}$ and $\tilde{w}_q := \tilde{w}_{\gamma_q}$. By the construction $\tilde{w}_0 = \tilde{w}$ and $\tilde{w}_1 = e$. So it is sufficient to show that $\tilde{w}_q = \tilde{w}_{q+1}$ for all $q, 0 \leq q \leq N$.

The loop $\gamma_{q+1} \subset X - Y$ is obtained from the e loop γ_q by crossing $Y_{\beta_q\beta'_q}$ which is a linear subspace of codimension 2 $[dim(Y_{\beta_q\beta'_q}) = r - 2]$. So we can think that γ_q consists of a curve γ_q^+ from t to a point s near $Y_{\beta_q\beta'_q}$ and a curve γ_q^- from s to t while the loop γ_{q+1} is obtained from γ_q by insertion of a small loop $\tilde{\gamma}$ which starts in s and goes around $Y_{\beta_q\beta'_q}$. But it is easy to see [please check] that

$$\tilde{w}_q^{-1}\tilde{w}_{q+1} = \tilde{w}_{\gamma_q^-}\tilde{w}_{\gamma'}\tilde{w}^{-1}_{\gamma_q^-}$$

Here γ' is a loop around $Y_{\alpha_q\alpha'_q}$ [as in the Example 2] where $(\alpha_q, \alpha'_q) = w^{-1}_{\gamma_q^+}(\beta_q, \beta'_q)$. Since we know [see Example 2)] that $\tilde{w}_\gamma = e$ we see that $\tilde{W} = e$ □