Lemma 0.1. *a) Let $S \subset T$ be a singular torus. There exists $\alpha \in X^\star$ such that $S = Ker^0(\alpha)$ and $Lie(Z_G(S)/R_u(Z_G(S))) = Lie(Z_G(T)) \oplus kX_\alpha \oplus kX_{-\alpha}$ where $X_\alpha, X_{-\alpha} \in \mathcal{G}$ are weight vectors for α and $-\alpha$ respectively.*

b) The subgroups $Z_G(S) \subset G$ where S runs through the set of singular subtori of T generate G.

Definition 0.2. a) Characters $\alpha \in X^\star$ as in Lemma 1 are *roots* of G.

b) We denote by $R = R(G,T) \subset X^\star$ the set of roots.

c) We define the action of the Weyl group $W = N_G(T)/T$ on $X^\star$ and $X_\star$ by

$$(wx^\star)(t) := x(w^{-1}t), (wx_\star)(a) := w(x_\star(a))$$

Lemma 0.3. *For any $\alpha \in R$ there exists unique $\alpha^\vee \in X_\star$ such that $< \alpha, \alpha^\vee >= 2$ and*

$$s_\alpha x = x - < x, \alpha^\vee > \alpha, s_\alpha x^\vee = x^\vee - < \alpha, x^\vee >, x \in X^\star, x^\vee \in X_\star$$

Definition 0.4. a) Elements $\alpha^\vee \in X_\star, \alpha \in R$ are *coroots* of G.

b) A root datum is a quadruple $\Psi = (X, R, X^\vee, R^\vee)$ where $X, X^\vee$ are finitely generated free abelian groups in duality $<,>: X \times X^\vee \to \mathbb{Z}, R \subset X, R^\vee \subset X^\vee$ finite subsets and an bijection $R \to R^\vee, \alpha \to \alpha^\vee$ such that $< \alpha, \alpha^\vee >= 2, \alpha \in R$ and $s_\alpha(R) = R, s_\alpha(R^\vee) = R^\vee$ where

$$s_\alpha x = x - < x, \alpha^\vee > \alpha, s_\alpha x^\vee = x^\vee - < \alpha, x^\vee >, x \in X^\star, x^\vee \in X_\star.$$

c) A root datum is *reduced* if for any $\alpha \in R, c \in \mathbb{R}$ such that $c\alpha \in R$ we have $c = \pm 1$.

d) We define $W(\Psi) \subset Aut(X)$ as the group W is generated by $s_\alpha, \alpha \in R$.

e) A subset $R^+ \subset R$ is *a system of positive roots* if $R = R^+ \cup -R^+$ and no nontrivial linear combination $\sum n_i \alpha_i, \alpha_i \in R^+, n_i \geq 0$ vanishes.

Lemma 0.5. *a) For any reductive group G the quadruple $\Psi = (X^\star, R, X_\star, R^\vee)$ is a reduced root datum and the group W is generated by $s_\alpha, \alpha \in R$.*

b) For any reduced root datum, system $R^+ \subset R$ of positive roots and any $\alpha, \beta \in R, \alpha \neq -\beta$ there exists $w \in W$ such that such that $w\alpha, w\beta \in R^+$.

c) There exists a positive definite W-invariant form $(,)$ on $X \otimes \mathbb{R}$. Using this form we identify X with $X^\vee$. It is clear that under this identification we identify $\alpha^\vee$ with $2(\alpha, \alpha)^{-1}\alpha$

d) For any $x \in X$ such that $(x,\alpha) \neq 0$ for all $\alpha \in R$ the set

$$R^+(x) := \{\alpha \in R | (x,\alpha) > 0\}$$

is a system of positive roots.

Proof of a). Given $w \in W := N_G(T)/T$ consider the homomorphism

$$\hat{w} : T \to T, t \to ntn^{-1}t^{-1}, t \in T$$

where $n \in N_G(T)$n is a representative of w. Then either

(i) $\hat{w}$ is not surjective or

(ii) There map $w - Id : X^\star_{\mathbb{R}} \to X^\star_{\mathbb{R}}$ is an isomorphism.

In the first case we replace G by $Z_G(S), S := Ker^0(\hat{w})$. In the second case choose any $\alpha \in R(G,T)$ and consider $x := (w - Id)^{-1}(\alpha) \in X^\star_{\mathbb{R}}$. Then

$$(x,x) = (wx,wx) = (x+\alpha, x+\alpha) = (x,x) + 2(x,\alpha) + (\alpha,\alpha)$$

So $< x, \alpha^\vee >= -1$ and therefore $(s_\alpha w)x = x$.□

Lemma 0.6. *Let $B \supset T$ be a Borel subgroup.*

a) For any singular torus $S \subset T$ there exists unique $\alpha_S \in R(G,T)$ such that $S = Ker^0(\alpha_S)$ and $Lie(B \cap Z_G(S)/R_u(Z_G(S))) = \mathcal{T} \oplus kX_\alpha$.

b) The set $R^+(B) := \{\alpha_S\}$ where S runs through the set of singular tori is a system of positive roots.

Proof of Lemma. a) We prove that for any root α either $\alpha \in R^+(B)$ or $-\alpha \in R^+(B)$ since $B \cap Z_G(S)$ is a Borel subgroup of $Z_G(S)$.

b) We choose a representation $\rho : G \to AutV$ and a vector $v \in V$ such that $B = St_G(v)$. Then the action of B of the line kv defines a character $x \in X^\star$. One shows that $< x, \alpha^\vee >> 0$ for all $\alpha \in R^+(B)$.□

Theorem 0.7. *Let G be a connected reductive group. Then the intersection $\tilde{V}$ of unipotent radicals of Borel subgroups containing T is finite.*

Let V be the identity component of $\tilde{V}$. It is sufficient to prove the V is a normal subgroup of G. For any root α we denote by H_α the identity component of intersection $\tilde{H}_\alpha$ of unipotent radicals of Borel subgroups containing T with $\alpha \in R^+(B)$.

Lemma 0.8. *V is a normal subgroup of H_α.*

Proof of Lemma. Use Lemma 5 b) to show that $Lie(H_\alpha) = Lie(V) \oplus kX_\alpha$. Therefore $dim(H_\alpha, V) = 1$.

Proof of Theorem. Since G is generated by $Z_G(S)$ where S runs through the set of singular tori and $Z_G(S)$ is generated by $Z_G(S) \cap B$ where B runs through Borel subgroups containing T the result follows from Lemma.

Corollary 0.9. *Let G be a connected reductive group. Then*

a) For any subtorus S the centralizer $Z_G(S)$ is connected and reductive.

b) $Z_G(T) = T$.

c) $Z(G) \subset T$

Proof of Corollary. a)

$$R_u(Z_G(S)) = \cap_{B \supset T} Z_G(S) \cap B_u \subset \cap_{B \supset T} B_u = R_u(G) = \{e\}$$

Lemma 0.10. *a) For any $\alpha \in R$ there exists an isomorphism $x_\alpha : \mathbb{G}_a \to X_\alpha$ where $X_\alpha \subset G$ is a closed subgroup such that*

$$tx_\alpha(a)t^{-1} = x_\alpha(\alpha(t)a), t \in T, a \in k$$

b) For any nontrivial homomorphism $x'_\alpha : \mathbb{G}_a \to G$ such that

$$tx'_\alpha(a)t^{-1} = x'_\alpha(\alpha(t)a), t \in T, a \in k$$

there exists $c \in k^\star$ such that $x'_\alpha(a) = x_\alpha(ca), a \in k$.

c) $Im(dx_\alpha) = \mathcal{G}_\alpha$.

d) T and $X_\alpha, \alpha \in R(G,T)$ generate G.

Corollary 0.11. *Roots of G are non-zero weights of the adjoint action $T \to Aut(\mathcal{G})$.*

Theorem 0.12. *Let G be a connected semisimple group. Then*

a) The subgroups $X_\alpha, \alpha \in R(G,T)$ generate G.

b) $G = (G,G)$.

c) Any connected normal subgroup $G_1 \subset G$ is semisimple and there exists a connected normal subgroup $G_2 \subset G$ such that $(G_1, G_2 = \{e\}, G_1 \cap G_2$ is finite and $G_1 G_2 = G$

Proof of Theorem. a) Consider the subgroup $H \subset G$ generated by $X_\alpha, \alpha \in R(G,T)$. As follows from Lemma 1 b) $H \subset G$ is a normal subgroup. Since $R(H) \subset R)G)$ we see that H is semisimple. By Lemma 1 b) the intersection $\cap_{\alpha \in R(G,T)} Ker(\alpha)$ lies in the center of G

we see that this group is finite. Therefore (?) roots $\alpha \in R(G,T)$ span a subgroup of finite index in X and the subgroup $\alpha^\vee(\mathbb{G}_m), \alpha \in R(G,T)$ span T. So $T \subset H$ and [by the same Lemma 1 b)] $H = G$.

b) Form the equality

$$tx_\alpha(a)t^{-1}x_\alpha(-a) = x_\alpha((\alpha(t)-1)(a))$$

it follows that $X_\alpha \in (G,G)$ for all $\alpha \in R(G,T)$.

c) Let $G_1 \subset G$ be a connected normal subgroup and $T_1 \subset T$ a maximal torus of T. Then

(i) using the equality as in b) you show that for any $\alpha \in R(G,T)$ we have $\alpha \in R_1 := R(G_1,T)$ iff $\alpha(T_1) \neq \{1\}$. We define $R_2 = R - R_1$.

(ii) We show that for any $\alpha \in R_1, \beta \in R_2$ the elements $x_\alpha(a), x_\beta(b) \in G$ commute for all $a, b \in k$. Really consider

$$x_b(a) := x_\beta(b)x_\alpha(a)x_\beta^{-1}(b)$$

It is clear that $x_b(a) \in G_1$ and $t_1x_b(a)t_1^{-1} = x_b(\alpha(t_1)a)$ for all $t_1 \in T_1$. By Lemma 10 b) there exists a homomorphism $f : \mathbb{G}_1 \to \mathbb{G}_m$ of algebraic groups such that $x_b(a) = x_\alpha(f(b)a)$. Since there is no nontrivial homomorphism $f : \mathbb{G}_1 \to \mathbb{G}_m$ of algebraic groups we see that $x_b(a) = x_\alpha(a)$. So the elements $x_\alpha(a), x_\beta(b) \in G$ commute. Let G_2 be the subgroup of G generated by $X_\alpha, \alpha \in R_2$. We see that $(G_1, G_2) = \{e\}$ and $G = G_1G_2$.