Problem 0.1. Let $\underline{f} : \underline{X} \to \underline{Y}$ be a morphism of algebraic varieties, $\underline{X}$ is affine and irreducible and $f : X \to Y$ is a bijection. Then $dim(X) = dim(Y)$

Definition 0.2. A *curve* is an algebraic variety of dimension 1.

Remark. I always assume that curves are irreducible.

Let $\underline{X}$ be an irreducible affine variety, $f : \underline{X} \to \underline{C}$ be a non-constant morphism to a curve, $c \in Im(f), Y := f^{-1}(c)$.

Claim. Then $dim(Y) = dim(X) - 1$.

This is a very useful result but the proof of this result is based on some results from Commutative algebra [such as the Normalization Lemma of Noether] but the proof requires more extensive knowledge of Algebra then I assume. I'll prove a very special case of the theorem which will suffice for our needs. We start with the following general result.

Lemma 0.3. *Let $\underline{X}$ be an irreducible affine variety, $f : \underline{X} \to \underline{C}$ be a non-constant morphism to a curve, There exists a curve $Y \subset X$ such that the restriction of f on Y is not a constant.*

Proof. It is clear (?) that we can assume that the curve C is affine. The proof is by induction in the dimension of X. If $dim(X) = 1$ then we can take $Y = X$. So assume that $dim(X) > 1$. It is sufficient to show the existence of a proper closed subset $Y \subset X$ such that that the restriction of f on Y is not a constant morphism.

Since $dim(X) > 1$ there exist (?) regular functions

$$g : X \to \mathbb{A}^1, r : C \to \mathbb{A}^1$$

such that $r \circ f$ and g are algebraically independent. Consider the map

$$\phi : X \to \mathbb{A}^2, x \to (r \circ f(x), g(x))$$

and define $X' := Im(\phi) \subset \mathbb{A}^2$. Then $X' \subset \mathbb{A}^2$ and since $\mathbb{A}^2$ is irreducible we see that $dim(\mathbb{A}^2 - X') < 2$. Therefore (?) there exists $b \in k$ such that the set $\{a \in k | (a, b) \notin X'\}$ is finite. Let $Y = g^{-1}(b)$. Then the restriction of f on Y is not a constant morphism.□

Lemma 0.4. *Let $f : \underline{X} \to \underline{C}$ be as Lemma 3 and assume that an algebraic group H acts on X without fixed points in such a way that fibers of f are H-orbits. Then $dim(H) = dim(G) - 1$.*

Proof.As follows from Problem 1 we have $dim(f^{-1}(x)) = dim(H)$ for all $x \in X$. Since X is irreducible we see that $dim(H) < dim(G)$. Assume that $dim(H) < dim(G) - 1$. Let $Y \subset G$ be a curve as in

Lemma 3 and consider the map $a : H \times Y \to G$ given by $a(h, y) := hy$. Since $f(Y)$ is not constant the image $Im(f)$ is dense in C and therefore the subset $Z := a(H \times Y)$ is dense in X. So

$$dim(H) + 1 = dim(H \times Y) \geq dim(Z) = dim(X)$$

So $dim(H) \geq dim(X) - 1.\square$

Problem 0.5. a) Let $\underline{f} : \underline{X} \to \underline{Y}$ be a morphism of irreducible algebraic varieties such that $f(X)$ is dense in Y.

a) There exists a closed subvariety Z of X such that $dim(Z) = dim(Y)$ and $f(Z)$ is dense in Y.

b) Assume that an algebraic group H acts on X without fixed points in such a way that fibers of f are H-orbits. Then $dim(H) = dim(X) - dim(Y)$.

Lemma 0.6. *Let $\underline{X}, \underline{Y}$ be irreducible algebraic varieties and $\underline{p} : \underline{X} \to \underline{Y}$ a morphism such that $p(X)$ is dense in Y and there exists an non-empty open subset U of Y such that $p^{-1}(u)$ is finite for all $u \in U$. Then $dim(\underline{X}) = dim(\underline{Y})$.*

Remark. The conclusion of the Lemma is true under the much weaker assumption. It is sufficient to know that there exists one $y \in Y$ such that the set $p^{-1}(y)$ is finite and not empty.

Proof. Since the image $p(X)$ is dense in Y it is easy to see (?) that $dim(\underline{X}) \geq dim(\underline{Y})$. Assume that $dim(X) > dim(Y)$. One can easily reduce the proof to the case when $\underline{X} = (X, A)$ and $\underline{Y} = (Y, B)$ are affine and the map $p : X \to Y$ is surjective. Since $p(X)$ is dense in Y we see that $p^\star : B \to A$ is an imbedding and we consider B as a subring of A.

Let F, E be the fields of fractions of the rings A, B. We want to show that $trdeg_k(B) = trdeg_k(A)$. Assume that $trdeg_k(B) < trdeg_k(A)$. Then we can find (?) $f \in A$ such that for any $b_0, ..b_n \in B, b_n \neq 0$ we have $\sum_{i=0}^{n} b_i f^i \neq 0$. Let $Z \subset Y \times k$ be the image of the map $x \to (p(x), f(x))$. Since Z is constructible we see (?) that Z is dense in $Y \times k$ and therefore there exists a closed proper subset $W \in Y \times k$ such that $Z \supset (Y \times k) - W$. Since Y is irreducible and W is proper subset of $Y \times k$ the intersection $(U \times k) \cap W$ is a proper subset of $U \times k$. Therefore (?) there exists $u \in U$ such that the intersection $(\{u\} \times k) \cap W$ is finite. But then the fiber $p^{-1}(u) = \{u\} \times k \cap Z$ is infinite.$\square$

Lemma 0.7. *a) If $P \subset G$ is a parabolic subgroup then for any action of $\underline{G}$ on $\underline{X}$ and a point $x \in X$ such that $P \subset St_x$ the orbit $\Omega(x) \subset X$ is closed and complete.*

b) A closed subgroup $P \subset G$ is parabolic iff there exists a finite dimensional representation $\rho : G \to GL(V)$ and a line $L \subset V$ such that $P = St_L$ and the orbit $\Omega(L) \subset \mathbb{P}(V)$ is closed.

c) If G is connected then $Z^0(G) \subset Z(B) \subset Z(G)$ where $Z(G)$ is the center of G and $B \subset G$ is a Borel subgroup.

d) If G is connected and B is nilpotent then $B = G$.

Proof of a). As follows from the proof of Proposition 14 there exists a finite dimensional representation $\rho : G \to GL(V)$ and a line $L \subset V$ such that $P \supset St_L$ and the orbit $\Omega(L) \subset \mathbb{P}(V)$ is closed. We can assume (?) $\Omega(x) \subset X$ is dense. Let $Y = \Omega(L)$. Consider the G orbit Z of the point $(x, L) \in X \times Y$ of diagonal action of G on $X \times Y$.

Claim 0.8. *$Z \subset X \times Y$ is closed.*

Proof of Claim. It is clear (?) that the restriction q of the projection $p_Y : X \times Y \to Y$ on Z is a bijection. Consider the closure $\bar{Z}$. Since Y is a G-orbit all the fibers of restriction $\bar{q}$ of the projection p_Y to $\bar{Z}$ are isomorphic. But this implies (?) that they consists of one point. Since $q : Z \to Y$ is onto we see that $\bar{Z} = Z$.□

Since Y is complete the projection $p_X(Z) \subset X$ is closed and proper [see Problem 5.3].□

Proof of b). It is easy to derive (?) from the part a) and the Chevalley theorem that for any for any parabolic subgroup $P \subset G$ there exists a finite dimensional representation and a line $L \subset V$ such that $P = St_L$ and the orbit $\Omega(L) \subset \mathbb{P}(V)$ is closed. Conversely, let $\rho : G \to GL(V)$ be Choose a Borel subgroup B'. As follows from Proposition 5.14 there exists a point $x \in \Omega(L)$ such that $B' \subset St_x$. Let $g \in G$ be such that $x = gL$. But then $B := g^{-1}B'g \subset P = St_L$.□

Proof of c). Since $Z^0(G)$ is connected and solvable it lies in some Borel subgroup B'. But since B and B' are conjugate $Z^0(G)$ lies in B. But then $Z^0(G) \subset Z(B)$.

To finish the proof of c) we have to show that any $z \in Z(B)$ belongs to $Z(G)$. Fix $z \in Z(B)$ and consider the morphism $f : \underline{G} \to \underline{G}, g \to gzg^{-1}$. Since $z \in Z(B), f$ factors through a morphism $\bar{f} : \overline{G/B} \to G$. Since $\underline{G/B}$ is complete and G is affine and $f(e) = z$ we see that $\bar{f} \equiv z$.□

Proof of d). The proof is by induction in $dim(B)$. If $B = \{e\}$ the $G = G/B$ is complete and affine. So $G = \{e\}$. If $B \neq \{e\}$ then [since B is nilpotent] $Z(B) \neq \{e\}$. Since $Z(B) \subset Z(G)$ we can replace G by $G/Z(B)$, B by $B/Z(B)$ and apply the inductive assumption.□

Definition 0.9. a) We denote the algebraic variety $\underline{G/H}(\rho_0, L)$ by $\underline{G/H}$ and call the natural morphism $\underline{\phi} : \underline{G} \to \underline{G/H}$ *the canonical projection.*

b) We define the *stabilizer* $St_{G/H} \subset G \times G/H$ of the action of G on G/H by $St_{G/H} := \{(g, x)|gx = x\}$ and write $X_{G/H} : p_G(St_{G/H}) \subset G$.

c) For any $h \in H$ we define $Y_h := \{y \in G/H|y^{-1}hy \in H\}$.

Problem 0.10. Show that

a) The stabilizer $St_{G/H}$ is a closed subset of $G \times G/H$.

b) $dim(St_{G/H}) = dim(G)$

[A hint] Use the result of Problem 5.

c) If $H \subset G$ is a parabolic subgroup then the image X of $St_{G/H}$ under the projection $G/H \times G \to G$ [$= \cup_{g\in G} gHg^{-1} \subset G$] is closed.

Lemma 0.11. *Assume that the there exists an open dense subset U of H such that the sets $Y_u, u\underline{U}$ are finite. Then $X = \cup_{g\in G} g^{-1}Hg$ is dense in G.*

Proof. Let $V := \cup_{g\in G, u\in U} g^{-1}ug \subset St_{G/H}$ and $\pi : V \to G$ be the restriction of the projection $p_G : G/H \times G \to G$ on $V, Z := \pi G(V)$. Since G is irreducible it is sufficient to show that $dim(Z) = dim(G)$.

Since for any $v = g^{-1}ug$ the fiber $\pi^{-1}(\pi(v)) = Y_u$ are finite it follows from Lemma 6 that $dim(Z) = dim(V) = dim(G) = dim(St_{G/H}) = dim(G)$.□

Solvable groups

Let T_n be the group of upper-triangular $n \times n$-matrices and $U_n \subset T_n$ the subgroup of unipotent upper-triangular matrices.

Problem 0.12. a) Let G be a closed connected subgroup of T_2. Then either $G = (e)$ or $G = U_2$ or G is conjugated to D_2 or $G = T_2$.

b) For any closed connected solvable group G the subset $G_u \subset G$ of unipotent elements is a closed normal subgroup of G.

Lemma 0.13. *a) Let $\underline{G}$ be a commutative connected affine algebraic group and G_s, G_u be the subsets of semisimple and unipotent elements. Then G_s, G_u are closed subgroups of G and $G = G_sG_u$.*

b) Let $\underline{G}$ be a solvable connected affine algebraic group such that all $g \in G$ are semisimple. Then $\underline{G}$ is diagonalizable. [that is $\underline{G}$ is isomorphic to a subgroup of the group $\underline{D}_n$ of diagonal matrices].

Proof of a). By the Levi-Kolchin Theorem we can assume that $G \subset T_n$. It is clear (?) that $G_u = G \cap U_n$. So $G_u \subset G$ is closed.

Since $\underline{G}$ be a commutative it is clear (?) that the subset $G_s \subset G$ is a subgroup. Moreover we can (?) choose a basis in k^n such that $G_s \subset D_n$ where $D_n \subset GL_n(k)$ is the subgroup of diagonal matrices. But then $\bar{G}_s \subset G \cap D_n = G_s$.□

Proof of b). As before we assume that $G \subset T_n$. Then the commutator $[G, G]$ lies in the subgroup U_n of unipotent upper-triangular matrices. Since all elements of G are semisimple we see that G is commutative. So we can choose a basis in k^n such that $G_s \subset D_n$.□

Theorem 0.14. *a) There exists a torus $\underline{T} \subset \underline{G}$ such that the map $T \times G_u \to G, (t, u) \to tu$ is one-to-one and onto,*

b) If $\underline{T}' \subset \underline{G}$ is any maximal torus in $\underline{G}$ then there exists $u \in G_u$ such that $uT'u^{-1} \subset T$.

Remark. One can show that the map $\underline{T} \times \underline{G}_u \to \underline{G}, (t, u) \to tu$ defines an isomorphism of algebraic varieties.

Proof. We will prove the Theorem by induction in $dim(G)$. So we assume that the result is known for all connected solvable groups $\underline{H}$ such that $dim(\underline{H}) < dim(\underline{G})$.

Since $\underline{G}$ is a solvable connected affine algebraic group we can assume that G is a subgroup of the group T_n. Let Λ be the set of pairs

$$\Lambda := \{(i, j)\}, 1 \leq i < j \leq n, \Lambda^* := \Lambda \cup \infty$$

We define an order on Λ^* by saying that $(i, j) < (p, q)$ if either

$j - i < q - p$ or $j - i = q - p$ and $i < p$

and say that $\infty > (i, j)$ for $, 1 \leq i, j \leq n$.

For any pair $(i, j), 1 \leq i, j \leq n$ we can consider the (i, j) matrix coefficient as a function

$a_{ij} : \underline{T}_n \to \mathbb{A}^1, a_{ij}(X) := x_{i,j}$ for $X = (x_{p,q}), 1 \leq p, q \leq n$

By the definition $a_{ij} = 0$ is $i > j$. For any subset $X \subset T_n$ we define $\Lambda(X) \subset \Lambda$ by

$$\Lambda(X) = \{\lambda \in \Lambda | a_\lambda(X) \neq 0\}$$

and define $\tilde{\lambda}(X) \in \Lambda$ by $\tilde{\lambda}(X) := min_{\lambda \in \Lambda(X)} \lambda$. So $\tilde{\lambda}(X) = \infty$ iff $X \subset D_n$. We define $\lambda(X) := max_{r \in T_n} \tilde{\lambda}(rXr^{-1})$. It is clear that it is sufficient to prove Theorem in the case when $\lambda(G) = \tilde{\lambda}(G)$. So we assume from now on that $\lambda(G) = \tilde{\lambda}(G)$.

Let $(i, j) = \lambda(G)$. Consider a map $\phi : G \to T_2$ given by

$$\phi(g) := \begin{pmatrix} a_{ii}(g) & a_{ij}(g) \\ 0 & a_{jj}(g) \end{pmatrix}$$

It is easy to see that the map $\phi : G \to T_2$ is homomorphism of algebraic groups.

Lemma 0.15. $Im(\phi) \supset U_2$

Proof of Lemma. If U_2 does not lie in $\bar{G}$ then it follows from Problem 7 that there exists $\bar{r} \in T_2$ such that $\bar{r}\bar{G}\bar{r}^{-1} \subset D_2$. Choose a preimage $r \in G$ of $\bar{r}$. Then (?) If we have $a_{ij}(rgr^{-1}) = 0$ for all $g \in G$ and therefore $\tilde{\lambda}(rGr^{-1}) < (i, j)$. But this contradicts the assumption that $\lambda(G) = \tilde{\lambda}(G)$. So $Im(\phi) \supset U_2$.□

Set $H := \phi^{-1}(D_2) \subset G$ and define

$$f : \underline{G} \to \mathbb{A}^1, f(g) := a_{ii}^{-1}(g)a_{ij}(g)$$

Since $f(hg) = f(g), h \in H, g \in G$ it follows from Lemma 4 that $dim(H) = dim(G) - 1(?)$. Let $\underline{H}^0 \subset \underline{H}$ be the connected component of $\underline{H}$ containing e.

Lemma 0.16. $H^0 G_u = G$

Proof. Since G_u is a normal subgroup of G the set $H^0 G_u$ is a subgroup of G. Since $G_u \subsetneq H^0$ and $dim(\underline{H}^0) = dim(\underline{G}) - 1$ we have $dim(\underline{H}^0 \underline{G}_u) = dim(\underline{G})$. Since $\underline{G}$ is connected we $H^0 G_u = G$.□

Now we can prove the part a) of the Theorem. By the construction $\underline{H}^0$ is a solvable connected affine algebraic group and $dim(\underline{H}^0) < dim(\underline{G})$. By the inductive assumptions there exists a torus $\underline{T} \subset \underline{H}$ such that the map the map $T \times H^0_u \to H^0, (t, u) \to tu$ is one-to-one and onto. Therefore the map $T \times G_u \to G, (t, u) \to tu$ is onto. Since $T \cap G_u = (e)$ the part a) of Theorem is proven. It is clear (?) that T is a maximal torus in G.

Now we prove the part b). Let $\underline{T}' \subset \underline{G}$ be a maximal torus. Consider $\underline{S}' := \phi(\underline{T}') \subset \underline{T}_2$. As follows from Problem 7 there exists $\bar{u} \in U_2$ such that $\bar{u}S'\bar{u}^{-1} \subset D_2$. By Lemma 10 there exists $u \in G_u$ such that $\bar{u} = \phi(u)$. Then $uT'u^{-1} \subset H$. The Theorem follows now from the inductive assumptions.□

Corollary 0.17. *Let $\underline{G}$ be a connected affine algebraic group. $\underline{T}, \underline{T}' \subset G$ be maximal tori. Then there exists $g \in G$ such that $gT'g^{-1} = T$.*

Proof. Let $\underline{B} \subset \underline{T}, \underline{B}' \subset \underline{T}' \subset \underline{G}$ be maximal connected solvable subgroups containing T and T'. By the Borel's theorem there exists $g \in G$ such that $gB'g^{-1} = B$. Then $g'T'g'^{-1} \subset B$. But [by the part b) of Theorem] the tori $T, g'T'g'^{-1} \subset B$ are conjugate in B.□

Let U be a connected unipotent normal subgroup of an algebraic group G and $s \in G$ a semisimple element. Define

$$\gamma_s(u) := usu^{-1}s^{-1}, u \in U, M := Im(\gamma_s), C := Z_U(s)$$

Problem 0.18. If $x \in Z(U), y \in U$ then $\gamma_s(xy) = \gamma_s(x)\gamma_s(y)$

Theorem 0.19. *The product morphism $\tau : C \times M \to U$ is bijective.*

Proof. To prove the injectivity of τ is it is sufficient to show that $C \cap M = \emptyset$. Choose any $c \in C \cap M$. Since $c \in M$ we have $c = usu^{-1}s^{-1}, u \in U$. Then $cs = usu^{-1}$. Since $c \in C$ the product cs is the Jordan decomposition of the semisimple element usu^{-1}. It follows from the uniqueness of Jordan decomposition that $c = e$.□

To prove the surjectivity of τ we consider first the case when U is commutative. Then [see Problem 18] τ and γ_s are group homomorphsims and $C = Ker(\gamma_s))$.

Since $\gamma_s : U \to M$ is a group surjective homomorphism we have $dim(U) = dim(C) + dim(M)$. It follows from the injectivity of τ that $dim(Im(\tau)) = dim(U)$. Since U is connected we see that τ is onto.□

We prove the general case by induction in $dim(U)$. Let $V := Z(U)^0$. Then V is a connected normal subgroup of $G, dim(V) > 0$. If $V = U$ then U is commutative and the result is already known. So we assume that $V \subsetneq U$. Define

$$G' := G/V, U' := U/V, s' := \pi(s) \in G'$$

where $\pi : G \to G'$ be the natural projection and denote by

$$\tau' : C' \times M' \to U', \tau_V : C_V \times M_V \to V$$

the maps corresponding to the triples $(G',', s')$ and (G, V, s). By the inductive assumptions we know that that maps τ' and τ_V are bijections.

Claim 0.20. *For any $c' \in C'$ there exists $c \in C$ such that $c' = \pi(c)$.*

Proof of Claim. Choose any $\tilde{c} \in U$ such that $c' = \pi(\tilde{c})$. Then we have $s\tilde{c}s^{-1} = \tilde{c}v, v \in V$. We want to find $y \in V$ such that $s(\tilde{c}y)s^{-1} =$

$\tilde{c}y$. For any $y \in V$ we have

$$s(\tilde{c}y)s^{-1} = s\tilde{c}s^{-1}sys^{-1} = \tilde{c}vsys^{-1} = \tilde{c}yy^{-1}vsys^{-1} = (\tilde{c}y)vsys^{-1}y^{-1}$$

since V is commutative. So we want to find $y \in V$ such that $vsys^{-1}y^{-1} = e$. As follows from the the surjectivity of τ_V we we can find $c \in U$ such that $\pi(c) = c'$ and $scs^{-1} = cz, z \in C_V$. But the $z \in C \cap M = \{e\}$.□

To prove the surjectivity of τ we have to show the existence of a decomposition $x = cm, c \in C, m \in M$ for any $x \in U$. Let $x' = \pi(x) \in U'$. As follows from the surjectivity of τ' we can write $x' = c'u', c' \in C', m' \in M'$. Let $c \in C$ be a preimage of c' as in the Claim. We have

$$x = c\tilde{v}usu^{-1}s^{-1}, u \in U, c \in C, \tilde{v} \in V$$

As follows from the surjectivity of τ_V we have

$$x = cc'vsv^{-1}s^{-1}usu^{-1}s^{-1}, u \in U, c, c' \in C, v \in V$$

Since $v \in Z(U)$ we have [see Problem 18] $x = (cc')\gamma_s(uv)$.□

Problem 0.21. The restriction of γ_s on M is bijective.

A hint. Use the following result. Let $\underline{f} : \underline{X} \to \underline{Y}$ be a morphism of algebraic varieties such that $f : X \to Y$ is a bijection. Then $f : X \to Y$ is a homeomorphism.

Corollary 0.22. *For any connected solvable group G and a semisimple $s \in G$ the centralizer $Z_G(s)$ is connected.*

Proof. As follows from Theorem 9 we have a decomposition $G = TG_u$ where T is a maximal torus of G containing s. Then $Z_G(s) = TZ_{G_u}(s)$. So it is sufficient to show that the group $Z_{G_u}(s)$ is connected. As follows from Theorem 14 we have a bijection $\tau : Z_{G_u}(s) \times M \to G_u$. So $Z_{G_u}(s)$ is connected. □