Definition 0.1. Let $\underline{G}$ be an affine group. A *character* of $\underline{G}$ is a group homomorphism $\chi : \underline{G} \to \mathbb{G}_m$. We denote by $X(\underline{G})$ [or $X(G)$] set of characters of $\underline{G}$. The set $X(\underline{G})$ has a group structure defined by

$$(\chi'\chi'')(g) := \chi'(g)\chi''(g)$$

Remind that we denoted by $D_n(k)$ the subgroup of diagonal matrices in $GL_n(k)$.

Lemma 0.2. *Let $\underline{G} = (G, A)$ be an affine group. The following three conditions are equivalent.*

a) G is commutative and all elements of G are semisimple.

b) $\underline{G} = (G, A)$ is isomorphic [as an affine group] to a subgroup of $D_n(k)$.

c) A is generated [as a vector space] by $\chi \in X(\underline{G})$.

Proof 0.3. I'll only outline the proof and leave for you to fill the details.

a) $\Rightarrow$ *b)*. As follows from Theorem 2.1 we can realize G as an algebraic subgroup of $GL_n(V)$ where V is a finite-dimensional vector space. If every $g \in G$ is a scalar matrix then there is nothing to prove. If G contains not-scalar matrices choose such $g \in G$. Since g is semisimple we have a decomposition of V as a direct sum $V = \oplus_\mu V_\mu$ such that $g_{|V_\mu} = \mu Id_{V_\mu}$. Since the subspaces V_μ are G-invariant (?) we finish by induction in $dim(V)$.

b) $\Rightarrow$ *c)*. Since $G \subset D_n(k)$ is a closed subset and the restriction of any character of $D_n(k)$ is a character of G it is sufficient to show that $k[\underline{D}_n]$ is generated by $\chi \in X(\underline{D}_n)$. Let $T_i, 1 \leq i \leq n$ be the function on $D_n(k)$ which associates to a diagonal matrix it $i-th$ diagonal entry. It is easy to see that $k[\underline{D}_n] = k[T_i^{\pm}], 1 \leq i \leq n$ and that the functions $\prod_{i=1}^n T_i^{n_i}, 1 \leq i \leq n, n_i \in \mathbb{Z}$ are characters of $D_n(k)$.

c) $\Rightarrow$ *a)*. We first show that G is commutative that is we show that $g_1 g_2 g_1^{-1} g_2^{-1} = e$ for all $g_1, g_2 \in G$. Let $g := g_1 g_2 g_1^{-1} g_2^{-1}$. It is clear that $\chi(g) = \chi(e) = 1$ for all $\chi \in X(G)$. Since A is generated by $\chi \in X(\underline{G})$ we see that $f(g) = f(e)$ for all $f \in A$. So $g = e$.

Let l be the left regular representation of G. To show that all elements of G are semisimple it is sufficient to show that for any $g \in G$ the locally finite operator $l(g) : A \to A$ is semisimple. But A is spanned by vectors $\{\chi\}, \chi \in X(G)$ which are eigen-vectors for $l(g)$.

Definition 0.4. An affine group is *diagonalizable* if it satisfies the conditions of this Lemma.

Problem 0.5. Show that

a) For any affine group G the characters $\chi \in X(G)$ are linearly independent.

b) For any diagonalizable affine group G the group $X(G)$ of characters is a finitely generated Abelian group and the order of the torsion part is prime to $ch(k)$. Moreover $X(G)$ is torsion free iff g is connected.

Sheaves

Definition 0.6. a) Let X be a topological space. A *presheaf* of sets $\mathcal{F}_X$ on X is a rule which

(i) associate to any open subset $U \subset X$ of X a set $\mathcal{F}(U)$ and

(ii) for any inclusion $U' \subset U$ of open subsets of X a map

$$r_{U,U'} : \mathcal{F}(U) \to \mathcal{F}(U')$$

such that $r_{U,U} = Id$ and

$$r_{U',U''} \circ r_{U,U'} = r_{U,U''}$$

for any triple $U'' \subset U' \subset U$ of open subsets.

b) we say that a presheaf $\mathcal{F}$ on X is a *sheaf* if for any open cover $U_i, i \in I$ of U and any $f_i \in \mathcal{F}(U_i), i \in I$ such that

$$r_{U_i,U_i \cap U_j}(f_i) = r_{U_j,U_i \cap U_j}(f_j), i, j \in I$$

there exists unique $f \in \mathcal{F}(U)$ such that $r_{U,U_i}(f) = f_i, i \in I$.

c) a (pre)sheaf of groups (algebras) on X is a (pre)sheaf $\mathcal{F}$ on X such that all the sets $\mathcal{F}_(U)$ have a group (algebra) structure and the restriction maps $r_{U,U'}$ are group (algebra) homomorphisms.

d) If $\mathcal{F}, \mathcal{F}'$ are (pre)sheaves on X. A *morphism* $a : \mathcal{F} \to \mathcal{F}'$ is a rule which associates to any open subset $U \subset X$ a map $a(U) : \mathcal{F}(U) \to \mathcal{F}'(U)$ such that for any inclusion $U' \subset U$ of open subsets of X we have $r_{U,U'} \circ a(U) = a(U') \circ r_{U,U'}$.

e) A subset $\mathcal{B}$ of open subsets of a topological space X is a *representative collection* of the topology on X if $X = \cup U, U \in \mathcal{B}$ and for any $U', U'' \in \mathcal{B}$.

f) Let $\mathcal{B}$ a representative collection of the topology on X. A $\mathcal{B}$-presheaf $\mathcal{F}_{\mathcal{B}}$ on X is a rule which

(i) associate to any open subset U in $\mathcal{B}$ a set $\mathcal{F}(U)$ and

(ii) for any inclusion $U' \subset U, U', U \in \mathcal{B}$ a map $r_{U,U'} : \mathcal{F}(U) \to \mathcal{F}(U')$ such that $r_{U,U} = Id$ and

$$r_{U',U''} \circ r_{U,U'} = r_{U,U''}$$

for any triple $U'' \subset U' \subset U, U'', U', U \in \mathcal{B}$.

g) a $\mathcal{B}$-presheaf $\mathcal{F}$ on X is a $\mathcal{B}$-*sheaf* if for any open cover $U_i, i \in I$ of $U, U_i, U \in \mathcal{B}$ and any $f_i \in \mathcal{F}(U_i), i \in I$ such that

$$r_{U_i, U_i \cap U_j}(f_i) = r_{U_j, U_i \cap U_j}(f_j), i, j \in I$$

there exists unique $f \in \mathcal{F}(U)$ such that $r_{U,U_i}(f) = f_i, i \in I$.

Problem 0.7. Let X be a topological space. Show that

a) For any $\mathcal{B}$-sheaf $\mathcal{F}_\mathcal{B}$ on X there exists a sheaf $\mathcal{F}$ on X and a family of isomorphisms $b(U) : \mathcal{F}_\mathcal{B}(U) \to \mathcal{F}_(U), U \in \mathcal{B}$ such that $r_{U,U'} \circ b(U) = b(U') \circ r_{U,U'}$ for any $U', U \in \mathcal{B}, U' \subset U$.

b) if $(\mathcal{F}', b'(U))$ is another such data then there exists unique isomorphism $a : \mathcal{F} \to \mathcal{F}'$ such that for any $U \in \mathcal{B}$ we have $b'(U) \circ a(U) = b(U)$.

In other words the sheaf $\mathcal{F}$ is defined uniquely up to the unique isomorphism.

c) Let $\mathcal{F}$ be a sheaf on X and Y be a closed subset of X. Then there exists a unique sheaf $\mathcal{F}_Y$ on Y such that for any open $U \subset X$ we have

$\mathcal{F}_Y(U \cap Y) =$ the restriction of $\mathcal{F}(U)$ on Y.

d) Let $\underline{X} = (X, A)$ be an affine variety. Show that set $\mathcal{B}$ of basic subsets $U_a \subset X, a \in A$ is a representative collection of the Zariski topology on X and that we have an inclusion $U_b \subset U_a, a, b \in A$ iff there exists $n \in \mathbb{N}, c \in A$ such that $b^n = ac$.

Definition 0.8. Let (X, A) be an affine algebraic variety and $\mathcal{B}$ the representative collection of the topology on X consisting of basic subsets $U_a, a \in A$.

For any A-module M and a basic subset $U_a \subset X$ we define the set

$$\mathcal{F}_M(U_a) := M_a, a \in A$$

where M_a is the localization of M in respect of the set $\{a^n\}, n \in \mathbb{N}$. Given $b, a \in A$ such that $U_b \subset U_a$ we define a map $r_{a,b} : M_a \to M_b$ by

$$r_{a,b}(m/a^d) := mc^d/b^{nd} \in M_b$$

where $b^n = ac$.

It is easy to see (?) that the image $r_{a,b}(m/a^d) \in M_b$ is well defined [that is does not depend on a choice of a decomposition $b^n = ac$] and that $(\mathcal{F}_M, r_{a,b})$ is a $\mathcal{B}$-presheaf on X [that is the conditions (i) and (ii) of the definition 6 are satisfied].

Main Theorem 0.9. *For any A-module M the $\mathcal{B}$-presheaf $\mathcal{F}_M$ is a $\mathcal{B}$-sheaf on X*

Proof 0.10. Proof. Let $U_{f_i}, f_i \in A, i \in I$ be a cover of X and $n_i \in M_{f_i}$ be such that $r_{f_i,f_if_j}(n_i) = r_{f_j,f_if_j}(n_j)$ for all $i, j \in I$. We have to show that there exists unique $m \in M$ such that n_i is the image of m in M_{f_i} for all $i \in I$.

Since the space X is quasi-compact we can assume that the set I is finite. Consider the ideal $(f_i) \subset A$ generated by $f_i, i \in I$. Since $U_{f_i}, i \in I$ is a cover of X there is no maximal ideal $\mathfrak{m}$ of A containing (f_i). So $(f_i) = A$. Therefore there exists $g_i \in A$ such that $\sum_{i \in I} f_i g_i = 1$.

First we prove the uniqueness of $m \in M$. Suppose we have two such elements $m', m'' \in M$. Let $n := m' - m''$. Then for any $i \in I$ the image of n in M_{f_i} is equal to 0. Therefore there exist $r_i \in \mathbb{N}$ such that $f_i^{r_i} n = 0$. Let $r := max_{i \in I} r_i$. Since the ideal generated by elements $f_i, i \in I$ is equal to A the ideal generated by elements $f_i^r, i \in I$ is also equal to $A(?)$. Therefore there exists $g_i' \in A, i \in I$ such that $\sum_{i \in I} g_i' f_i^r = 1$. Then we have

$$n = (\sum_{i \in I} g_i' f_i^r) n = \sum_{i \in I} g_i' f_i^r n = 0$$

I'll prove the existence in the case when $I = (1, 2)$ and leave for you to extend the proof to the general case. We have to show that for any a_1, a_2 such that the ideal (a_1, a_2) is equal to A and $m_1/a_1^q \in M_{a_1}, m_2/a_2^q \in M_{a_2}$ such that $(m_1 a_2^q - m_1 a_2^q)(a_1 a_2)^p = 0, p >> 0$ there exists $m \in M$ such that $(m_1 - a_1^q m) a_1^r = (m_1 - a_1^q m) a_1^r = 0$ for $r >> 0$. By replacing a_1, a_2 by their powers we can assume that we have $m_1/a_1 \in M_{a_1}, m_2/a_2 \in M_{a_2}$ such that $(a_2 m_1 - a_1 m_2)(a_1 a_2) = 0$.

Claim 0.11. *There exists $m' \in M$ such that $(m_1 - a_1 m')a_1 = 0$.*

Choose $b_1, b_2 \in A$ such that $a_1 b_1 + a_2 b_2 = 1$ and write

$$m_1 = a_1 b_1 m_1 + a_2 b_2 m_1 = a_1 b_1 m_1 + b_2 a_1 m_2 + b_2 c, n := a_2 m_1 - a_1 m_2$$

But $n = a_1 b_1 n + a_2 b_2 n$ and therefore $m_1 = a_1 m' + d$ where $m' := b_1 m_1 + b_2 m_2 + b_2 b_1 n, d := a_2 b_2 n$. Since $a_1 d = 0$ we see that $(m_1 - a_1 m')a_1 = 0$.□

Analogously you can find $m'' \in M$ such that $(m_2 - a_2 m'')a_2 = 0$. Now we can replace m_1/a_1 by m' and m_2/a_2 by m''. So we have to show that for any $m', m'' \in M$ such that $(m' - m'')(a_1 a_2) = 0$ there exists

$m \in M$ such that $(m - m')a_1 = (m - m'')a_2 = 0$. Since $m' - m'' = a_1b_1(m' - m'') + a_2b_2(m' - m'')$ can take

$$m := m' - a_2b_2(m' - m'') = m'' + a_1b_1(m' - m'')\square$$

Definition 0.12. a) As follows from Problem 7 the $\mathcal{B}$-sheaf $\mathcal{F}_M$ on X defines a sheaf on X which we also denote by $\mathcal{F}_M$.

b) The sheaf $\mathcal{O}_X := \mathcal{F}_A$ is called the *structure sheaf* of (X, A). It has a natural structure of a sheaf of rings.

b) The sheaves on $\mathcal{F}$ on X of the form $\mathcal{F}_M$ where M is an A-module are called *quasi-coherent* sheaves. They have a natural structure of a sheaves of $\mathcal{O}_X$-modules [that is for any open $U \subset X, \mathcal{F}_M(U)$ has a natural structure of an $\mathcal{O}_X(U)$-module].

Algebraic varieties

Definition 0.13. a) *An algebraic prevariety* is a pair $\underline{X} = (X, \mathcal{O}_X)$ where X is a quasi-compact topological space, $\mathcal{O}_X$ is a sheaf of rings functions on X with values in k such that for any $x \in X$ there exists on open subset $U \subset X, x \in U$ such that the pair $(U, \mathcal{O}_X(U))$ is isomorphic to an affine algebraic variety.

b) An algebraic prevariety $(X, \mathcal{O}_X)$ is an *an algebraic variety* iff the diagonal [=image of $\Delta : X \to X \times X$] is closed in $X \times X$.

c) Let $\underline{X} = (X, \mathcal{O}_X), \underline{Y} = (Y, \mathcal{O}_Y)$ be algebraic varieties. A morphism from $\underline{X}$ to $\underline{Y}$ is a continuous map $\phi : X \to Y$ such that for any open subset U of Y and any $f \in \mathcal{O}_Y(U)$ the function $\phi^\star(f) : V \to k, V := f^{-1}(U)$ given by $\phi^\star(f) := f(\phi(x)$ belongs to $\mathcal{O}_X(V)$.

Remark 0.14. As follows from Theorem 9 any affine algebraic variety is an algebraic prevariety.

Problem 0.15. a) Show that

a) Any affine algebraic variety $\underline{X}$ is an algebraic variety which we also denote by $\underline{X}$. Moreover for any affine algebraic varieties $\underline{X}, \underline{Y}$ the set of morphisms from $\underline{X}$ to $\underline{Y}$ as algebraic varieties is the same as the set of morphisms from $\underline{X}$ to $\underline{Y}$ as affine algebraic varieties.

b) Let $\underline{X} = (X, \mathcal{O}_X)$ be an algebraic variety and $Y \subset X$ a closed subset. Then the pair $Y, \mathcal{O}_Y$ where $\mathcal{O}_Y := \mathcal{O}_{XY}$ is an algebraic variety.

c) If $(X, \mathcal{O}_X)$ is an algebraic variety and $U, V \subset X$ are open affine then $U \cap V$ is also open affine and the images of $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$ in $\mathcal{O}_X(U \cap V)$ generate $\mathcal{O}_X(U \cap V)$ as a subalgebra.

d) If $(X, \mathcal{O}_X)$ is an algebraic prevariety then X is a Noetherian topological space.

e) Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be algebraic prevarieties. Show that there exists unique structure of an algebraic prevariety on the set $X \times Y$ such that for any open affine subsets $U \subset X, V \subset Y$ the subset $U \times V \subset X \times Y$ is an open affine.

f) for any algebraic prevariety X the diagonal map $\Delta : X \to X \times X$ is a homeomorphism of topological spaces.

g) *. Let X be an algebraic prevariety, $U_i, i \in I$ an open cover of X by open affine. Then X is an algebraic variety iff for any $i, j, 1 \leq i, j \leq m$ the intersection $U_i \cap U_j$ is affine and the algebra $\mathcal{O}_X(U_i \cap U_j)$ is generated by $r_{U_i, U_i \cap U_j}(\mathcal{O}_X(U_i)$ and $r_{U_j, U_i \cap U_j}(\mathcal{O}_X(U_j))$.

h) * Give an example of a sheaf of $\mathcal{O}_X$-modules over the line $\mathbb{A}^1 = (K, K[t])$ which is not a quasi-coherent sheaf.

Projective space $\underline{\mathbb{P}}^n$.

Definition 0.16. As a set $\mathbb{P}^n(k)$ is the set of all lines $L \subset k^{n+1}$. Equivalently $\mathbb{P}^n(k) = k^{n+1} - \{0\}/\sim$ where the equivalence relation $\sim$ is given by $x \sim y$ iff $x = \lambda y, \lambda \in k^*$.

Let $q : k^{n+1} - \{0\} \to \mathbb{P}^n(k)$ be the natural projection. We write $x^* = q(x) \in \mathbb{P}^n(k)$ for the equivalence class of $x = (x_0, ..., x_n) \in k^{n+1} - \{0\}$ and call $\{x_i\}, 0 \leq i \leq n$ the *homogeneous coordinates* of x^*.

For any $i, 0 \leq i \leq n$ we put

$$U_i := \{(x_0, ..., x_n)^* \in \mathbb{P}^n(k) | x_i \neq 0\}$$

and define bijections $\phi_i : U_i \to k^n$ by

$$\phi_i(x_0, ..., x_n) := (x_i^{-1}x_0, ..., x_i^{-1}x_{i-1}, x_i^{-1}x_{i+1}, ..., x_i^{-1}x_n)$$

using the maps ϕ_i we will identify U_i with $\mathbb{A}^n$.

We define a topology on $\mathbb{P}^n(k)$ by saying that a set $U \subset \mathbb{P}^n$ is open iff for any $i, 0 \leq i \leq n$ the set $\phi_i(U \cap U_i) \subset U_i$ is open.

We define a presheaf $\mathcal{O}_{\mathbb{P}^n}$ on $\mathbb{P}^n$ by defining

$$\mathcal{O}_{\mathbb{P}^n}(U) := \{f : U \to k | f_{U_i} \in \mathcal{O}_{U \cap U_i}, \forall i, 0 \leq i \leq n\}$$

For any subset Z of $\mathbb{P}^n(k)$ we define $Z^* := 0 \cup q^{-1}(Z) \subset k^{n+1}$ and $\mathcal{I}^*(Z) := \mathcal{I}(Z^*) \subset k[x_i], 0 \leq i \leq n$.

Problem 0.17. Show that

a) $\mathcal{O}_{\mathbb{P}^n}$ is a sheaf on $\mathbb{P}^n(k)$.

b) $\underline{\mathbb{P}}^n := (\mathbb{P}^n(k), \mathcal{O}_{\mathbb{P}^n})$ is an algebraic variety such that ϕ_i is a isomorphism between $(U_i, \mathcal{O}_{\mathbb{P}^n|U_i})$ and $\mathbb{A}^n$.

c) Any linear automorphism g of k^{n+1} defines an automorphism of $\underline{\mathbb{P}}^n$ [which we will also denote by g].

d) Let Z be a subset of $\mathbb{P}^n(k)$. Then Z is closed iff Z^* is closed and Z is irreducible iff Z^* is irreducible.

e) The ideal $\mathcal{I}^*(Z)$ is homogeneous [that is $\mathcal{I}^*(Z)$ is generated by homogeneous polynomials].

f) For any radical homogeneous proper ideal $I^* \subset k[x_i]$ there exist unique closed subset $Z(I^*) \subset \mathbb{P}^n(k)$ such that $Z^*(I^*) = \mathcal{V}(I^*)$.

g) $Z(I^*) = \emptyset$ iff $I^* = \{f \in k[x_0, ..., x_n] | f(0) = 0\}$.

i)* Let $\underline{Y} = (Y, A)$ be an affine algebraic variety. Describe a relation between homogeneous ideals of $A[x_i], 0 \leq i \leq n$ and closed subvarieties of $\mathbb{P}^n(k) \times Y$.

j) Let V be a finite-dimensional k-vector space. Define an algebraic variety $\underline{\mathbb{P}}(V) = (\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)})$ such that $\mathbb{P}(V)$ is the set of lines in V.

h) Consider a map $f : \mathbb{P}^n(k) \times \mathbb{P}^m(k) \to \mathbb{P}^{mn+m+n}(k)$ given by

$$f((x_0, ..., x_n)^*, (y_0, ..., y_m)^*) := (x_i y_j)^*$$

Show the image $V^{n,m} \subset \mathbb{P}^{mn+m+n}(k)$ of f is closed and f defines an isomorphism $f : \mathbb{P}^n(k) \times \mathbb{P}^m(k) \to V^{n,m}$.

k) $V^{1,1} \subset \mathbb{P}^3$ is given by one homogeneous equation [that is the homogeneous ideal $\mathcal{I}^*(V^{1,1}) \subset k[x_0, ..., x_3]$ is principal]. Find this equation.

Definition 0.18. Let V be an n-dimensional k-vector space. For any $m < n$ we denote by $\Lambda^m(V)$ the m-exterior power of V, by $Gr_m(V)(k)$ the set of m-dimensional subspaces L of $V(k)$ and by $\phi_m : Gr_m(V) \to \mathbb{P}(\Lambda^m(V))$ a map given by

$$\phi_m(L) := \Lambda^m(i_L)(\Lambda^m L) \subset \mathbb{P}(\Lambda^m(V))$$

where $i_L : L \to V$ is the natural imbedding.

Let $e_i, 1 \leq i \leq n$ be a basis of V. By the definition the vector space $\Lambda^m(V)$ has a basis $e_{\bar{i}}, \bar{i} \in I$ where I is the set subsets of $[1, n]$ of size m. For any $\bar{i} \in I$ we denote by $p_{\bar{i}} : V \to V_{\bar{i}}$ the natural projection where $V_{\bar{i}}$ is the subspace spanned by $e_i, i \in \bar{i}$. For any $\bar{i} \in I$ we denote the subset $U_{\bar{i}} \in Gr_m(V)$ of subspaces $W \subset V, dim(W) = m$ such that restriction of the projection $p_{\bar{i}}$ on W defines an isomorphism $p_{\bar{i}} : W \to V_{\bar{i}}$.

We denote by $\mathcal{B}(V)(k)$ the set of complete flags $W_1 \subset W_2 \subset ... \subset W_n = V$ in V where W_i is an i-dimensional subspace of V and by

$$\kappa : \mathcal{B}(V)(k) \to \prod_{m=1}^{n-1} Gr_m(V)(k), \kappa(W_1 \subset W_2 \subset ... \subset W_d) = (W_1, W_2, ..., W_{n-1}$$

Problem 0.19. a) Construct a "natural" bijection $U_{\bar{i}} \to k^d$ [please find d] and define a structure of an algebraic prevarity on $Gr_m(V)$.

Show that

b) $\cup_{\bar{i} \in I} U_{\bar{i}} = Gr_m(V)$.

c) The image $\phi(Gr_m(V)) \subset \mathbb{P}(\Lambda^m(V))$ is closed and ϕ defines an isomorphism of $Gr_m(V)$ with the image of ϕ in $\mathbb{P}(\Lambda^m(V))$.

d) The image $\kappa(\mathcal{B}(V)(k))$ in $\prod_{m=1}^{n-1} Gr_m(V)(k)$ is closed and κ defines an isomorphism of an algebraic variety $\underline{\mathcal{B}}(V)$ with the image of κ in $\prod_{m=1}^{n-1} \underline{Gr}_m(V)$.

e) In the case when $n = 4, m = 2$ the image $Z := \phi(Gr_2(V)) \subset \mathbb{P}(\Lambda^2(V))$ is closed is defined by one homogeneous quadratic equation. Please find this equation.

f)* Find the system of quadratic equations for the image $\phi(Gr_m(V)) \subset \mathbb{P}(\Lambda^m(V))$.

g)** Find the system of quadratic equations for the image κ in $\prod_{m=1}^{n-1} \mathbb{P}(\Lambda^m(V))$.

In the next problem you construct and study an important class of sheaves on the projective space $\mathbb{P}^n$ called *Line bundles* .

Problem 0.20. a) Show that there exists unique sheaf $\mathcal{O}(r)$ and isomorphisms

$$\phi_i : \mathcal{O}(r)_{U_i} \to \mathcal{O}_{U_i}, 0 \leq i \leq n$$

such that for any $i, j \in [0, n]$ the map

$$\phi_{i U_i \cap U_j} \circ \phi_{j}{}^{-1}_{U_i \cap U_j} : \mathcal{O}_{U_i \cap U_j} \to \mathcal{O}_{U_i \cap U_j}$$

is given by the multiplication by $(x_i/x_j)^r$.

b) For any linear automorphism g of k^{n+1} construct an isomorphism $g_* : \mathcal{O}(r) \to g^*(\mathcal{O}(r))$ [see the definition 5.1] in such a way that for any two linear automorphisms g', g'' of k^{n+1} we have $(g'g'')_* = g'_* g''_*$.

c) Evaluating g_* on $\mathcal{O}(r)(\mathbb{P}^n)$ we obtain a representation of the group $GL(n+1, k) = Aut(k^{n+1})$ on the space $\mathcal{O}(r)(\mathbb{P}^n)$.

d) Show that for any open subset $U \subset \mathbb{P}^n$ we have $\mathcal{O}(r)(U) = \{f \in \mathcal{O}(q^{-1}(U)|f(\lambda x) = \lambda^r f(x)\}, \lambda \in k^*$.

e) Find $dim(\mathcal{O}(r)(\mathbb{P}^n))$ and describe the representation of the group $GL(n+1, k) = Aut(k^{n+1})$ on the space $\mathcal{O}(r)(\mathbb{P}^n)$.