I'll start with a reminder of some basic results from Commutative algebra. Let A be a commutative ring. We say that the ring A is *Noetherian* if any increasing sequence of ideals in A stabilizes. For any ideal I of A we define the *radical* $\sqrt{I}$ of I as the ideal of all the elements $a \in A$ such that $a^n \in A$ for some $n > 0$. Let K be an algebraically closed field.

Definition 0. Let A be a commutative ring with 1. Given an element $f \in A$ we define the localization of A with respect to f to be the ring the quotient ring $A_f := A[t]/ft$ and denote by $\phi_f : A \to A_f$ the algebra morphism induced by the imbedding $A \hookrightarrow A[t]$.□

Problem 1. (a) Show that we can define the ring A_f as a set of fractions $a/f^k, k \in \mathbb{N}$ modulo the following equivalence relation:

$a/f^k \equiv b/f^l$ iff $f^N(f^l a - f^k b) = 0$ for some $N \in \mathbb{N}$.

(b) Describe the kernel of the canonical morphism $\phi_f : A \to A_f$. In what cases the localized ring A_f is trivial, i.e. consists of one element 0 ?

(c) Show that if A has no nilpotents (resp. no zero divisors), then A_f has no nilpotents (resp. no zero divisors).□

Let K be an algebraically closed field. $A := K[T_1, ..., T_n]$ the ring of polynomials in $T_1, ..., T_n$. For any $x = (x_1, \ldots, x_n) \in K^n$ we denote by $I_x \subset A$ the maximal ideal generated by $T_i - x_i, 1 \leq i \leq n$ and by $ev_x : A \to K$ the algebra homomorphism such that $ev_x(T_i) = x_i, 1 \leq i \leq n$. For any ideal I of A we define

$\mathcal{V}(I) := \{x \in K^n | ev_x(f) = 0 \text{ for all } f \in I\}$.

Conversely for any subset Y of K^n we define

$I(Y) := \{f \in A | f(y) = 0 \text{ for all } y \in Y\}$.

Claim. a) The ring A is *Noetherian*.

b) *Nullstellensatz*. Any maximal ideal $I \subset A$ is equal to I_x for some $x \in K^n$.

c) For any ideal I of A we have $\sqrt{I} = I(\mathcal{V}(I))$.□

Definition 1. a) For any set X we denote by $K\{X\}$ the K-algebra of K-valued functions on X. For any $x \in X$ we denote by

$$ev_x : K\{X\} \to K$$

the homomorphism of the evaluation at x.

b) We say that a commutative K-algebra A is *reduced* if any nilpotent element of A is equal to 0,

c) A K-*affine variety* is a pair (X, A) where X is a set and A is a is finitely generated reduced K-subalgebra of $K\{X\}$ such that for any K-homomorphism $\phi : A \to K$ there exists unique $x \in X$ such that $\phi = ev_x$. I often write an *affine variety* instead of a K-*affine variety.*

d) if $I \subset A$ is an ideal we define
$\mathcal{Z}(I) := \{x \in X | f(x) = 0 \text{ for all } f \in I\}$

e) if $S \subset X$ we define $\mathcal{I}(S) := \{f \in A | f(s) = 0 \text{ for all } s \in S\}$

f) We call the subalgebra $A \subset K\{X\}$ the algebra of *regular functions* on X.

g) Let (X, A) be an affine variety. For any $f \in A$ we define

$$U_f := \{x \in X | f(x) \neq 0\}, S := f^m, m \in \mathbb{N}\square$$

Remark. Since the ring A if finitely generated over a field K it follows from **Claim** a) that A is *Noetherian.*

Example 1. Let $X = K^n$. For any $i, 1 \leq i \leq n$ denote by
$T_i : K^n \to K$ the function given by $T_i(x_1, ..., x_n) := x_i$ and define $A_n \subset K\{K^n\}$ to be the ring $K[T_1, ..., T_n]$ of polynomials in $T_1, ..., T_n$. Then (K^n, A_n) is an affine variety which we denoted $\mathbb{A}^n$.

Lemma 1. Let (X, A) be an affine variety. Then

i) For any two ideals I', I'' of A we have $\mathcal{Z}(I') \cup \mathcal{Z}(I'') = \mathcal{Z}(I' \cap I'')$

ii) For any family $\{I_b, b \in B\}$ of ideals in A we have

$$\cap_{b \in B} \mathcal{Z}(I_b) = \mathcal{Z}(\sum_{b \in B} I_b)$$

iii) For any $f \in A$ the pair (U_f, A_f) is also an affine variety.$\square$

Problem 2. Prove Lemma 1.

Definition 2. Let (X, A) be an affine variety. The *Zariski topology* on X is the topology such that a subset $Y \subset X$ is closed iff $Y = \mathcal{Z}(I)$ for some ideal $I \subset A$.

Remarks a) Lemma 1 shows that the *Zariski topology* is a topology on X.

b) In this course all affine varieties are furnished with the Zariski topology.$\square$

Definition 3. Let $\underline{X} = (X, A), \underline{Y} = (Y, B)$ be an affine varieties. We say that a map $\phi : X \to Y$ is a *morphism* from $\underline{X}$ to $\underline{Y}$ if for any $h \in B$ we have $h \circ \phi \in A$. In this case we denote by $\phi^\star : B \to A$ the homomorphism given by $h \to h \circ \phi$. We demote by $Mor(\underline{X}, \underline{Y})$ as the set of morphisms from $\underline{X}$ to $\underline{Y}$.

We say that a morphism $\phi : X \to Y$ is an *isomorphism* of affine varieties is ϕ is bijective and $\phi^{-1} : Y \to X$ is also a morphism of affine varieties.

Remark Sometimes I write a morphism from $\underline{X}$ to $\underline{Y}$ as a pair $(\phi, \phi^\star)$.

Problem 3. a) Let $(X, A), (Y, B)$ and (Z, C) be affine varieties and $\phi : X \to Y, \psi : Y \to Z$ be morphisms of affine varieties. Show that $\psi \circ \phi : X \to Z$ is a morphism of affine varieties.

Let A be a finitely generated K-algebra with no nilpotents define the set $X = Mor_{K-alg}(A; k)$.

b) Show that A embeds into the algebra $K\{X\}$ of K-valued functions on X and that the pair (X, A) is an affine algebraic variety.

(c) Let $\underline{X} = (X, A), \underline{Y} = (Y, B)$ be affine algebraic varieties. Show that the map $(\phi, \phi^\star)$ defines a bijection between the set $Mor(\underline{X}; \underline{Y})$ and the set $Mor_{K-alg}(B, A)$ of K-algebra morphisms from B to A.

d) Describe all closed subsets of $K = K^1$.

e) Let V be a finite-dimensional K-vector space and A the ring of K-valued polynomial functions on V. Show that $\underline{V} := (V, A)$ is an affine algebraic variety. We say that $\underline{V}$ is an affine variety associated with the vector space V.

f) . Let $\phi : X \to Y$ be a morphism of affine varieties such that , the associated algebra homomorphism $\phi^* : K[Y] \to K[X]$ is surjective. Then ϕ is an imbedding and $Im(\phi)$ is a closed subset of Y.$\square$

Definition 4. Let A be a commutative algebra and $I \subset A$ be an ideal. We say that $I \subset A$ is *primitive* iff $\sqrt{I} = I$

Problem 4. Let (X, A) be an affine variety.

a) The map $\mathcal{I} : S \to \mathcal{I}(S)$ defines a bijection between closed subsets of X and primitive ideals of A. Moreover the map $\mathcal{Z} : I \to \mathcal{Z}(I)$ is the inverse of $\mathcal{I}$

b) Let $I, J \subset A$ be primitive ideals. Then $J \subset I$ iff $\mathcal{Z}(I) \subset \mathcal{Z}(J)$

c) For any primitive ideal $I \subset K[T_1, ..., T_n]$ the pair $(\mathcal{V}(I), K[T_1, ..., T_n]/I$ is an an affine variety.

d) Formulate the analog of c) for any affine variety (X, A).□

Definition 5. A topological space is *Noetherian* if for any decreasing sequence $F_1 \supset F_2 \cdots \supset F_i \ldots$ of closed subsets there exists $j \in \mathbb{N}$ such that $F_k = F_j$ for all $k \geq j$

Definition 6. A topological space is quasi-compact iff from any covering of X by open sets one can choose a finite subcovering.

Problem 5. Let (X, A) be an affine variety. Show that

a) any point $x \in X$ is closed,

b)X is a Noetherian quasi-compact topological space,

c) any polynomial functions f on X the map $f : X \to K$ is continuous [we consider K as a topological space in the Zariski topology].□

Problem 6. Let (X, A) be an affine variety, $Y \subset X$ a closed subset, $I := \mathcal{I}(Y)$. Show that the pair $(Y, A/I)$ is an affine variety.

Definition 7. Let A be a commutative K-algebra with a unit and $S \subset A$ a subset containing the unit $1_A \in A$ which is closed under the multiplication.

A localization of A_S of A in respect to S is a pair $(A_S, \phi : A \to A_S)$ where A_S is a K-algebra and $\phi : A \to A_S$ is an K-algebra homomorphism such that

(i) $\phi(s)$ is invertible for all $s \in S$ and

(ii) for any K-algebra homomorphism $\psi : A \to B$ such that $\psi(s)$ is invertible for all $s \in S$ there exists unique K-algebra homomorphism $\kappa : A_S \to B$ such that $\psi = \kappa \circ \phi$.□

Problem 7. a) Show that the pair $(A_f, \phi_f : A \to A_f)$ is the localization of A in respect to the set $(1, f, f^2, \ldots, f^n, \ldots)$

b) Let $(A'_S, \phi' : A \to A'_S), (A''_S, \phi'' : A \to A''_S)$ be localizations of A. Show that there exists unique K-algebra homomorphism $\kappa : A' \to A''$ such that $\phi'' = \kappa \circ \phi'$ and that κ is an algebra isomorphism.

Remark . We can say that b) shows that the localization of A in respect S is unique up to a unique isomorphism.

c*) Show the existence of the localization $(A_S, \phi : A \to A_S)$ for any multiplicative subset S of A.□

Definition 8. a) Let A, B be K-vector spaces and V be the K-vector space with a base $a \otimes b, a \in A, b \in B$. Let $V_0 \subset V$ be the subspace spanned by the vectors

i) $(a' + a'') \otimes b - a' \otimes b - a'' \otimes b$

ii') $a \otimes (b' + b'') - a \otimes b' - a \otimes b''$,
iii') $k(a \otimes b) - (a \otimes kb), a \in A, b \in B, k \in K$

iii") $k(a \otimes b) - (ka \otimes b), a \in A, b \in B, k \in K$

We denote by $A \otimes B$ the quotient space V/V_0. By the construction $A \otimes B$ is a K-vector space.□

Problem 8. Show that

a) $dim(A \otimes B) = dim(A)dim(B)$ if A, B are finite-dimensional K-vector spaces.

Let A, B be commutative K-algebras. Then

b) there exists unique commutative K-algebra structure on $A \otimes B$ such that $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ for all $a, a' \in A, b, b' \in B$.

c*) if algebras A and B are reduced then the algebra $A \otimes B$ is also reduced .

d) For any affine varieties $(X, A), (Y, B)$ the map

$a \otimes b \to a(x)b(y)$

defines an isomorphism of the K-algebra $A \otimes B$ onto a subalgebra of $K\{X \times Y\}$ and that the pair $(X \times Y, A \otimes B)$ is an affine variety.

e) Construct an isomorphism $\mathbb{A}^n \times \mathbb{A}^m \to \mathbb{A}^{m+n}$.□

Problem 9. Let $I := (f) \subset K[x, y]$ be the ideal generated by $f = x^2 - y^3$.

a) Show that I is primitive.

b) Let $\underline{Z} = (A, Z)$ be the affine subvariety of $\mathbb{A}^2$ corresponding to I [so $A = K[x, y]/I, Z = \{(x, y) \in K^2 | x^2 = y^3\}$]. Construct a morphism $(f, f^\star) : \mathbb{A}^1 \to \underline{Z}$ such that $f : K \to X$ is a bijection and show that it is a homeomorphism of topological spaces [in the Zariski topology].

c*) Prove that the affine varieties $\underline{Z}$ and $\mathbb{A}^1$ are not isomorphic.