One of the formulations of the quantum mechanics and the quantum field theory is in terms of *Feynman integrals* which are integrals over an infinite-dimensional manifold X of *fields*. To define these integrals we have to define rigorously a measure on X. So we start with a discussion of some elements of the measure theory on real vector spaces.

Definition 1. We say that a function f on a vector space V is *positive definite* if the following three conditions are satisfied.

a) $f(0) = 1$.

b) $f(-v) \equiv \overline{f(v)}, v \in V$ where for a complex number z we denote by $\bar{z}$ the complex conjugate of z.

c) For any $n > 0$ and any n-tuple $\bar{v} = \{v_1, ..., v_n\}, v_i \in V$ the Hermitian $n \times n$ matrix $a_{ij}(\bar{w}) := f(w_i - w_j)$ is non-negative.

Let V be a locally convex topological vector space and μ be a normalized measure on the dual vector space $V^\vee$ [that is $\int_{V^\vee} \mu = 1$]. We define the *Fourier transform* of the measure μ as the function $\tilde{\mu}$ on the dual space V given by

$$\tilde{\mu}(v) := \int_{V^\vee} \exp(\sqrt{-1} < \phi, v >)\mu(\phi), v \in V$$

Lemma 1. The Fourier transform of any normalized measure μ on $V^\vee$ is positive definite.

Proof. It is clear that $\tilde{\mu}$ satisfies the conditions a) and b) of the Definition 1. To prove the part c) one has to show that for any n complex numbers $z_i, 1 \leq i \leq n$ and n vectors $v_1, ..., v_n$ we have $\sum_{i=1}^n \tilde{\mu}(v_i - v_j)z_i\bar{z}_j \geq 0$. But by the definition we have

$$\sum_{i=1}^n \tilde{\mu}(v_i - v_j)z_i\bar{z}_j = \int_V |\sum_{i=1}^n z_i \exp(\sqrt{-1} < \phi, v_i >)|^2\mu(\phi)$$

Theorem 1. (Bochner) Any positive definite function f on a finite-dimensional vector space V is the Fourier transform of the unique normalized measure on $V^\vee$.

You can find a proof in Rudin, W. (1990), Fourier analysis on groups, Wiley-Interscience.

Definition 2. Let V be a vector space. We say that a linear operator $A : V \to V^\vee$ a *positive-definite* if the quadratic form $Q_A(v) :=< A(v), v > /2$ on V is positive-definite.

Example. Let V be a finite-dimensional vector space and $A : V \to V^\vee$ be a positive-definite linear operator. In this case the inverse operator $A^{-1} : V^\vee \to V = (V^\vee)^\vee$ is also positive-definite and we can consider the positive-definite quadratic form $Q_{A^{-1}}$ on $V^\vee$. We denote by μ_A the measure on $V^\vee$ of the form $\mu_A = c_A \exp(-A^{-1}(v))dv^\vee$ where $dv^\vee$ is an invariant Lebesgue measure on $V^\vee$ and the constant $c_A \in \mathbb{R}_+$ is such that the measure μ_A is normalized.

Lemma 2. $\tilde{\mu}_A = \exp(-Q_A)$.

Proof. By the definitions we have $\tilde{\mu}_A(v) = \int_{V^\vee} c_A \exp(-Q_A(\phi) + \sqrt{-1} < \phi, v >)d\phi$ for $v \in V$. Let $\phi_0 := A(v) \in V^\vee$. Then

$$-Q_A(\phi) + \sqrt{-1} < \phi, v >= -Q_A(\phi + \sqrt{-1}\phi_0) - Q_A(v)$$

and after the change of variables $\phi \to \phi + \sqrt{-1}\phi_0$ we see that $\tilde{\mu}_A(v) = \exp(-Q_A)(v)$.

Please justify these arguments!!!

Definition 3. For any normalized measure μ on a vector space $V^\vee$ we denote by $W_n(\mu)$ the *correlation function* on V^n given by $\phi_1, ..\phi_n \in V^\vee$

$$W_n(v_1, .., v_n) := \int_{V^\vee} < \phi, v_1 >< \phi, v_2 > ... < \phi, v_n > \mu(\phi)$$

[under the assumption of the convergence].

Corollary. For finite-dimensional vector space V and a positive-definite linear map $A : V \to V^\vee$ we have

$W_{2n+1}(\mu_A) \equiv 0, W_2(\mu_A)(v_1, v_2) = G_A(v_1, v_2)$ where $G_A(v_1, v_2) :=< v_1, A^{-1}(v_2) >$ and $W_{2n}(\mu_A)(v_1, .., v_{2n}) = \sum_\sigma W^\sigma_{2n}(\mu_A)(v_1, .., v_{2n})$ where

$$W^\sigma_{2n}(\mu_A)(v_1, .., v_{2n}) := \prod_{i\in[1,2n]/\sigma} W_2(\mu_A)(v_i, v_{\sigma(i)})$$

where σ runs through the set of free involutions on the set $[1, 2n]$. In other words σ runs through the set of parings on $[1, 2n]$ and $W^\sigma_{2n}(\mu_A)(v_1, .., v_{2n})$ is the product of the corresponding pairwise correlations.

Proof. Since the function $< \phi, v_1 >< \phi, v_2 > ... < \phi, v_{2n+1} >$ on $V^\vee$ is odd and the measure μ_A is invariant under $\phi \to -\phi$ we see that $W_{2n+1}(\mu_A) \equiv 0$. To compute $W_2(\mu_A)(v_1, v_2)$ we consider the value of second derivative $\partial^2/\partial v_1 \partial v_2$ of the both sides of the equality $\int_{V^\vee} c_A \exp(-Q_A(v) + \sqrt{-1} < \phi, v >)dv = \exp(-Q_{A^{-1}})$ at $\phi = 0$. To find $W_{2n}(\mu_A)(v_1, .., v_{2n})$ one has to apply the $2n$-th derivative $\partial^{2n}/\partial v_1 \partial v_2 ... \partial v_{2n}$ to the both sides of the same equality and to evaluate the result at 0.

It is clear that using this Corollary we can compute the integral $\int_{V^\vee} P(\phi)\mu_A$ for any polynomial function P on $V^\vee$. Therefore we can also compute correlation functions of measures μ of the form $\mu = \mu_A \exp(\epsilon P(\phi))$ for any polynomial function P on $V^\vee$ as formal power series in ϵ. The combinatorics of these computations is performed in terms of *Feynman graphs* to be discussed later.

Now let's consider the infinite-dimensional case. Let now V be a locally convex topological vector space and f be a positive-definite function on V. One can asked when is it a Fourier transform of a measure on $V^\vee$.

Theorem 2 (Minlos). If V is a nuclear space then any positive-definite function on V is a Fourier transform of the unique measure on $V^\vee$.

You can find on my web page a definition and basic properties of nuclear spaces. I will not repeat now the definition but will give a couple of examples.

a) For any compact C^∞-manifold M the space $\mathcal{S}(M)$ of smooth functions on M is a nuclear space.

b) The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing smooth functions on $\mathbb{R}^n$ is a nuclear space.

Let now V be a nuclear space and $A : V \to V^\vee$ be a positive-definite linear operator. Then the function $\exp(-Q_A(v))$ is positive definite and by the theorem of Minlos is a Fourier transform of a measure μ_A on $V^\vee$ which is called a *Gaussian* *measure*. It is easy to see that the correlation function of the measure μ_A are given by the same formulas as in the finite-dimensional case.

Consider the case when M is a compact Riemannian C^∞-manifold, $V = \mathcal{S}(M)$ and $A = (-\Delta+1)^{-1}$ where Δ is the Laplacian. The corresponding measures μ_A on the space $\mathcal{D}(M)$ of distributions correspond to free field theories and majority of rigorous works on Quantum Field Theories are on the study of formal perturbations of such measures.

Let's analyze the corresponding correlation functions. Fix two points $m_1 \neq m_2$ on M consider

$$W_2(m_1, m_2) := \int_{\mathcal{D}(M)} \phi(m_1)\phi(m_2)\mu_A(\phi)$$

Since elements of $\mathcal{D}(M)$ are distributions the linear functionals $\phi \to \phi(m), m \in M$ are not well defined but it is easy to see that the correlation functions $W_2(m_1, m_2)$ are well defined and could be computed as limits of $W_2(v_1^n, v_2^n)$ where $v_1^n, v_2^n \in \mathcal{S}(M)$ are any sequences convergent to $\delta_{m_1}, \delta_{m_2}$ such that $supp(v_1^n) \cap supp(v_2^m) = \emptyset$. Moreover one can easily check that $W_2(m_1, m_2) = G(m_1, m_2)$ where $G(m_1, m_2)$ the *Green function* of the operator $-\Delta + 1$ [that is for any $m_1 \in M, G(m_1, m)$ is a distribution on M such that $(-\Delta + 1)(G(m_1, m) = \delta_{m_1}]$. In particular we see that the functions $G(m_1, m_2)$ on $M \times M - \Delta_M$ [where Δ_M is the diagonal] extend to continuous functions on $M \times M$ if $dim(M) = 1$, have logarithmic singularities at Δ_M when $dim(M) = 2$ and have singularities at Δ_M of the form $d^{dim(M)-2}(m_1, m_2)$ when $dim(M) > 2$ where $d(m_1, m_2)$ is the distance on M between points m_1 and m_1.

For the computation of Feynman integrals one has to consider integrals of the form $\int_{\mathcal{D}(M)} \int_M P(\phi)\mu_A(\phi)$ which are expressed [using the combinatorics of Feynman graphs] in term of $\int_{M\times M} G(m_1, m_2)dm_1dm_2$. In the case when $dim(M) \leq 2$ the Green functions are integrable and the integrals of the form $\int_{\mathcal{D}(M)} \int_M P(\phi)\mu_A(\phi)$ are well defined. Therefore one can easily define the formal perturbations μ_ϵ of the measure μ_A of the form $\exp(\epsilon \int_M R(\phi)dm)\mu_A$ and to compute the correlation functions of μ_ϵ for any polynomial R. But in the case when $dim(M) > 2$ the definition of the formal perturbations μ_ϵ is quite tricky. In this course we discuss the contemporary approach to the perturbative quantum field theory.