Let A be the algebra generated by p, q, z with the relations $[p,q] = z, [p,z] = [q,z] = 0$.

Claim 0.1. (1) $p^a q^b = \sum_{k=0}^{a} \binom{a}{k} \prod_{j=0}^{a-k-1} (b-j) q^{b-a+k} p^k z^{a-k}$
(2) $p^{2b-a} q^{b+a} p^{2a-b} = \sum_{i=0}^{b+a} c(a,b,i) p^{a+b-i} q^{a+b-i} z^i$ *where* $c(a,b,i)$ *are polynomials of degree* i.

Definition 0.2. Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a semi-simple Lie algebra.
(1) We fix a complete order on the set $\alpha \in \Phi^+$ of positive roots.
(2) We denote by σ the anti-involution of $\mathfrak{g}$ such that

$$\sigma_{|\mathfrak{h}} = Id, \sigma(x_\alpha) = y_\alpha, \sigma(y_\alpha) = x_\alpha \in \Phi$$

(3) We denote by $q : U(\mathfrak{g}) \to U(\mathfrak{h})$ the projection defined by the isomorphism $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n})$.
(4) Let A be the bilinear form on $U(\mathfrak{g})$ with values in $U(\mathfrak{h}) = S(\mathfrak{h})$ given by $A(x,y) := q(\sigma(x)y), x, y \in U(\mathfrak{g})$
(5) For any $\eta \in \Lambda_+$ we denote by A_η the restriction of the form A to $U(\mathfrak{n}^-)_\eta$ and by $D_\eta \in S(\mathfrak{h})$ the discriminant of the form A_η which is defined uniquely up to a multiplication by $c \in \mathbb{C}^\star$.
(6) A *partition* ω of $\eta \in \Lambda_+$ is an presentation of η as a sum $\eta = \alpha_1 + \alpha_2 + ... + \alpha_d(\omega)$ where α_i are positive roots such that $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_d(\omega)$. We denote by $\tilde{\mathcal{P}}(\eta)$ the set of partition of η.
(7) For any partition ω of $\eta \in \Lambda_+$ we define $y_\omega := y_{\alpha_1} y_{\alpha_2} ... y_{\alpha_d} \in U(\mathfrak{n}^-)_{-\eta}$

Claim 0.3. (1) $|\tilde{\mathcal{P}}(\eta)| = \mathcal{P}(\eta)$.
(2) $deg(A_\eta(y_{\omega'}, y_{\omega''})) \leq min(d(\omega'), d(\omega''))$.
(3) *If* $\omega' \neq \omega''$ *and* $d(\omega') = d(\omega'')$ *then* $deg(A_\eta(y_{\omega'}, y_{\omega''}) < d(\omega')$.
(4) *If* $\omega' = \omega''$ *then* $deg(A_\eta(y_{\omega'}, y_{\omega''})) = d(\omega)$.
(5) $deg(D_\eta) = \sum_{\omega \in \tilde{\mathcal{P}}(\eta)} d(\omega)$.

Theorem 0.4 (Shapovalov). $D_\eta = \prod_{\alpha \in \Phi^+} \prod_{r>0} (h_\alpha + \rho(\alpha) - r)^{\mathcal{P}(\eta - r\alpha)}$

Proof. Let $\tilde{D}_\eta := \prod_{\alpha \in \Phi^+} \prod_{r>0} (h_\alpha + \rho(\alpha) - r)^{\mathcal{P}(\eta - r\alpha)}$. As follows from the previous Claim we have $deg(D_\eta) = deg(\tilde{D}_\eta)$. So it is sufficient to prove that D_η is divisible by $(h_\alpha + \rho(\alpha) - r)^{\mathcal{P}(\eta - r\alpha)}$ for any $\alpha \in \Phi^+, r > 0$.

Given $\alpha \in \Phi^+, r > 0$ we choose a generic λ on the hyperplane $< \alpha, \lambda + \rho >= r$. Since we know the existence of an imbedding $M(\lambda - r\alpha) \hookrightarrow M(\lambda)$ we see that the kernel of the canonical map $\phi : M(\lambda) \to M(\lambda)^\vee$ contains $M(\lambda - r\alpha)$. Therefore $D_\eta(\lambda + t\rho)$ is divisible by $t^{\mathcal{P}(\eta - r\alpha)}$ in the ring $\mathbb{C}[[t]]$. □

Remark 0.5. One can prove the Shapovalov's theorem and therefore the Jantzen's formula without the knowledge of the existence of imbeddings $M(\lambda - r\alpha) \hookrightarrow M(\lambda)$ [see Kac, V. G.; Kazhdan, D. A. Structure of representations with highest weight of infinite-dimensional Lie algebras. Adv. in Math. 34 (1979), no. 1, 97108.] and then deduce the existence of imbeddings $M(\lambda - r\alpha) \hookrightarrow M(\lambda)$ from this formula.

The relative Lie algebra cohomology. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{b}$ a Lie subalgebra and $y \to \bar{y}$ be the projection from $\mathfrak{g}$ to $\mathfrak{g}/\mathfrak{b}$ We denote by D_k the $U(\mathfrak{g})$-modules $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \Lambda^k(\mathfrak{g}/\mathfrak{b})$ and define differentials $d_k : D_k \to D_{k-1}$ for $k > 0$ by

$$d_k(u \otimes x_1 \wedge ... \wedge x_k) = \sum_{i=1}^{k} (-1)^{i+1} u y_i \otimes x_1 \wedge ... \wedge \hat{x}_i \wedge ... \wedge x_k) +$$
$$+ \sum_{1 \leq i < j \leq k} (-1)^{i+j} u \otimes \overline{[y_i, y_j]} \wedge x_1 ... \wedge \hat{x}_i \wedge ... \wedge x_j \wedge ... \wedge x_k)$$

where $\bar{y}_i = x_i$ and define $d_0 : D_0 \to \mathbb{C}$ by $d_0(u \otimes 1) = \epsilon(u)$ where $\epsilon : U(\mathfrak{g}) \to \mathbb{C}$ is the counit.

LEMMA 0.6. (1) *The differential d_k is well defined [that is the right side does not depend on a choice of preimages y_i of x_i].*
(2) $d_{k-1} \circ d_k = 0$ *for all* $k > 0$
(3) *The complex* $... \to D_k \to .... \to D_0 \to \mathbb{C}$ *is exact. In other words the complex* $... \to D_k \to .... \to D_0$ *is a resolution of the* $U(\mathfrak{g})$-*module* $\mathbb{C}$.

REMARK 0.7. If $dim(\mathfrak{g}) = n < \infty$ then $D_k = \{0\}$ for $k > n$.

PROOF. The proof is pretty standard. I'll outline a proof only in the case when $\mathfrak{g}$ is the Lie algebra of a Lie group G and $\mathfrak{b}$ is the Lie algebra of a subgroup B of G. Let Ω^* be the De-Rham complex on G/B and $\hat{\Omega}^*$ be the completion of Ω^* at $e \in G/B$. The natural action of G on G/B induces an action of the Lie algebra $\mathfrak{g}$ on $\hat{\Omega}^*$. I'll leave for you to construct an identification of D_k with continuous linear functional on $\hat{\Omega}^k$ in such a way that the differential d_k is dual to the De-Rham differential on $\hat{\Omega}^*$. The Lemma follows now from the exactness of the De-Rham complex. □

DEFINITION 0.8. Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a semi-simple Lie algebra.
(1) We denote by $-R_k \subset \mathfrak{h}^\vee$ the set of weights in the decomposition of $\mathfrak{h}$-module $\Lambda^k(\mathfrak{n}^-)$. So
$$\Lambda^k(\mathfrak{n}^-) = \sum_{\mu \in R_k} \Lambda^k(\mathfrak{n}^-)_{-\mu}$$
(2) For any $w \in W$ we denote by $\Phi(w) \subset \Phi^+$ the subset of positive roots γ such that $w(\gamma) \in \Phi^-$ and write $\mu(w) := \sum_{\gamma \in \Phi(w)} \gamma$.

CLAIM 0.9. (1) $\mu(w) \in R_{l(w)}$ *for any* $w \in W$ *and the space* $\Lambda^{l(w)}(\mathfrak{n}^-)_{-\mu(w)}$ *is one-dimensional.*
(2) *For any* $w', w'' \in W$ *such that* $l(w') = l(w'')$ *and any* $\gamma \in \Phi^+$ *we have* $\mu(w') - \mu(w'') - \gamma \notin \Lambda^+$.