Definition 0.1. Let $A$ be a unital finite-dimensional algebra over an algebraically closed field $k$ and $M_i, 1 \leq i \leq r$ be representatives of non-isomorphic simple $A$-modules.

(1) We define the *radical* of $A$ by

$$Rad(A) := \{a \in A | \pi(a) = 0\}$$

for all irreducible representations $\pi$ of $A$.

(2) A finite-dimensional algebra $A$ is *semisimple* iff $Rad(A) = \{0\}$.

Claim 0.2. (1) *$Rad(A) \subset A$ is a two-sided nilpotent ideal.*

(2) *If the algebra $A$ is semisimple then the maps $A \to End_k(M_i), 1 \leq i \leq r$ induce an isomorphism $A \to \oplus_{i=1}^r End_k(M_i)$.*

Lemma 0.3. *Let $A$ be a ring, $I \subset A$ a two-sided nilpotent ideal, $\bar{A} := A/I$ and $\bar{e} \in \bar{A}$ an idempotent. [That is $\bar{e}^2 = \bar{e}$]. Then*

(1) *There exists a lift of $\bar{e}$ to an idempotent $e \in A$*

(2) *Any two such such lifts are conjugate by an element in $Id + I$.*

Proof. By induction it is easy to reduce the proof to the case when $I^2 = \{0\}$. So we assume that $I^2 = \{0\}$. In this case $I$ has a structure of a two-sided $A/I$-module. Let $\tilde{e} \in A$ be any lift of $\bar{e}$ and $a := \tilde{e}^2 - \tilde{e} \in I$. Any lift $e$ of $\bar{e}$ has the form $e = \tilde{e} + b, b \in I$ and the condition $e^2 = e$ is equivalent to the condition $\bar{e}b + b\bar{e} - b = a$. For the proof of the first claim it is sufficient to note that $b := (2\bar{e} - 1)a$ satisfies this condition. Let $e'$ be another lift of $\bar{e}$ such that ${e'}^2 = e'$. Then $e' = e + c$ where $c \in I$ is such that $ec + ce = c$. Since $e^2 = e$ this equation implies that $ece = 0$ and that $(1 - e)c(1 - e) = 0$. So $c = ec(1 - e) + (1 - e)ce = [e, [e, c]]$. Hence [since $I^2 = \{0\}$] we have $e' = (1 + [c, e])e(1 + [c, e])^{-1}$. □

Definition 0.4. A complete system of orthogonal idempotents in a unital algebra $B$ is a collection of elements $e_1, ..., e_n \in B$ such that

$$e_i e_j = e_i \delta_{i,j}, 1 \leq i, j \leq n$$

Corollary 0.5. *Given a complete system of orthogonal idempotents $\bar{e}_1, ..., \bar{e}_n \in A/I$ there exists lift $e_1, ..., e_n \in A$ of $\bar{e}_1, ..., \bar{e}_n$ to a complete system of orthogonal idempotents in $A$.*

Proof. The proof is by induction in $m$. If $m = 2$ then this Corollary is a restatement of the previous Lemma. For $m > 2$ we choose a lift $e_1$ of $\bar{e}_1$ and apply the inductive assumption to the algebra $(1 - e_1)A(1 - e_1)$. □

Theorem 0.6. (1) *For any $i, 1 \leq i \leq r$ there exists unique indecomposable finitely generated projective $A$-module $P_i$ such that*

$$dim(Hom_A(P_i, M_j)) = \delta_{i,j}$$

(2) *$A = \oplus_{i=1}^r d_i P_i$ where $d_i := dim_k(M_i)$.*

(3) *Any indecomposable finitely generated projective $A$-module is isomorphic to $P_i$ for some $i, 1 \leq i \leq r$.*

Proof. For any $i, 1 \leq i \leq r$ choose a basis $\{m_t^i\}, 1 \leq t \leq d_i$ of $M_i$ and denote by $\bar{e}_t^i \subset End(M_i)$ the projection on the line $km_t^i$ along the hyperplane generated by vectors $m_s^i, s \neq t$. As we know $Rad(A) \subset A$ is a two-sided nilpotent ideal and $A/Rad(A) = \oplus_{i=1}^r End_k(M_i)$ and it is clear that $\{\bar{e}_t^i\}$ is a complete system of orthogonal idempotents in $A/Rad(A) = \oplus_{i=1}^r End_k(M_i)$. Let $\{e_t^i\} \in A$ be a lift of $\{\bar{e}_t^i\}$ to a complete system of orthogonal idempotents in $A$. We define $P_{i,t} := Ae_t^i \subset A$ for $1 \leq i \leq r, 1 \leq t \leq d_i$. Then $A = \oplus_{1 \leq i \leq r, 1 \leq t \leq d_i} P_{i,t}$ and we see that the $A$-modules $P_{i,t}$ are projective.

By the construction $Hom_A(P_{i,t}, M_j) = e_t^i M_j$. So we see that $dim(Hom_A(P_{i,t}, M_j)) = \delta_{i,j}$. Since for a fixed $i$ the elements $\bar{e}_t^i \in End(M_i)$ are conjugated by an element of $End^\star(M_i)$ it follows form Lemma 1 that the elements $e_t^i \in A, 1 \leq t \leq d_i$ are conjugated by an element of $A^\star$ and therefore the $A$-modules $P_{i,t}, 1 \leq t \leq d_i$ are isomorphic. We will write $P_i$ instead of $P_{i,t}, 1 \leq t \leq d_i$.

I'll leave for you to check that the modules $P_i$ are indecomposable and that any indecomposable finitely generated projective $A$-module is isomorphic to $P_i$ for some $i, 1 \leq i \leq r$. □

Definition 0.7. Let $\mathcal{C}$ be an abelian $k$-category

(1) We say that $\mathcal{C}$ is *finite* if
   (a) It has a finite number of equivalence classes of simple objects $M_i, 1 \leq i \leq r$ and $End_{\mathcal{C}}(M_i) = k$ for all $i, 1 \leq i \leq r$.
   (b) Every object of $\mathcal{C}$ has finite length and
   (c) For any simple object $M \in Ob(\mathcal{C})$ there exists a projective object $P \in Ob(\mathcal{C})$ such that $Hom_{\mathcal{C}}(P, M) \neq \{0\}$.

(2) We say that a projective object $P \in Ob(\mathcal{C})$ is a *progenerator* if any object of $\mathcal{C}$ is a quotient of some finite multiple of $P$.

Problem 0.8. Let $\mathcal{C}$ be a finite abelian $k$-category. Then

(1) A projective object $P \in Ob(\mathcal{C})$ is a progenerator if $Hom_{\mathcal{C}}(P, M) \neq \{0\}$ for any simple object $M \in Ob(\mathcal{C})$
(2) There exists a progenerator $P \in Ob(\mathcal{C})$

Definition 0.9. Let $\mathcal{C}$ be a finite abelian $k$-category, $P \in Ob(\mathcal{C})$ be a progenerator. We denote

(1) by $A_P$ the ring $End_{\mathcal{C}}(P)^{op}$
(2) by $A_P - fmodules$ the category of finitely generated $A_P$-modules
(3) by $F_P$ the functor from $\mathcal{C}$ to the category $A_P - fmodules$ given by

$$F_P(M) := Hom_{\mathcal{C}}(P, M)$$

Theorem 0.10. *The functor $F_P(M)$ defines an equivalence between $\mathcal{C}$ and the category $A_P - fmodules$.*

PROOF. The action of $A_P$ on $P$ defines a map $\alpha_P \in Hom_{\mathcal{C}}(P \otimes_k A_P, P)$. For any finitely generated $A_P$-module $X$ we define $\alpha_X \in Hom_{A_P}(A_P \otimes_k X, X)$ as the action map from $(a\otimes \to ax, a \in A_P, x \in X$. We denote by $G$ the functor from the category $A_P - fmodules$ to $\mathcal{C}$ given by $G(X) =: Coker(\psi_X)$ where $\psi_X \in Hom_{\mathcal{C}}(P \otimes_k A_P \otimes_k X, P \otimes_k X)$ is given by

$$\psi_X := \alpha_P \otimes Id_X - Id_P \otimes \alpha_X : P \otimes_k A_P \otimes_k X \to P \otimes_k X$$

I'll leave for you to construct isomorphisms $F \circ G \to Id_{A_P - fmodules}$ and $G \circ F \to Id_{\mathcal{C}}$ □