Social choice theory

Topics in Discrete Mathematics: CS/Math 945

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1 Week 1: Introduction

Social choice theory studies the way individual preferences aggregate into the choice of the society. It is thus originated in economics and political science. When "individuals" refer to processors in a distributed computer or to agents on the Internet we quickly reach areas of interest to computer science.

1.1 Condorcet

Why is social choice related at all to mathematics? A pioneer in using mathematics for "political science" was Condorcet (1743-1794). I will describe now two famous discoveries of Condorcet.

A. The Condorcet's paradox.

It is possible that the majority of the society prefers alternative A to alternative B and prefers alternative B to alternative C and still prefers alternative C to alternative A.

B. Condorcet's "Jury Theorem".

Suppose there are two candidates Alice and Bob. Alice is the best candidate and every voter gets a signal "Alice is better!" with probability p and a signal "Bob is better!" with probability $1-p$. $p > 1/2$ is a fixed real number. The signals are statistically independent. As the number of voters tends to infinity with probability tending to 1 the majority of voters will prefer Alice.

We will come back to these results and their interpretation later on.

1.2 Models of individual choice and preferences and the notion of rationality

What I describe below can be found (and much more) in standard text books in “microeconomic”. E.g.,Lectures one-three in “Lecture notes in Microeconomic theory” by Ariel Rubinstein available at: `http://arielrubinstein.tau.ac.il/Rubinstein`

Model I: Preference relations.

A preference relation: For every pair of alternatives a and b the individual either prefers a to b, denoted by $a \succ b$ or prefers b to a (denoted by $b \succ a$) or is indifferent between a and b, denoted by $a \sim b$.

We will call a preference relation strict if there are no cases of indifference.

A preference relation is called transitive if (I) $a \sim b$ and $b \sim c$ implies $a \sim c$,

and (II) $a \succ b$ and $b \succ c$ implies $a \succ c$.

The axiom of rationality: rational preference relations are transitive.

Discussion:

1) What is fun in this topic is that there is a lot to discuss. Does the notions and models appropriate. Do they carry some hidden assumptions? What is the interpretation of the results?

2) The limitation of the model.

Preference relation do not allow us to express things like “I prefer a much more than b” or, “I prefer a to b now but will prefer b to a tomorrow,” or “I prefer the alternative I have now,” or I prefer the first alternaitve”.

3) Does the axiom of rationality really expresses rationality?

There are various critique about this notion of rationality both from the descriptive and normative points of view. One critique is that this notion of rationality is too restricted. Another critique is that this notion is too inclusive.

We continue with models of individual preferences and choice.

Model II: Utility functions.

Every alternative a is assigned a real number $U(a)$ called the "utility of a".

For two alternatives a and b, $a \succ b$ if $U(a) > U(b)$ and $a \sim b$ if $U(a) = U(b)$.

Model III: Choice functions.

Given a set X of n alternatives a choice function is a map c which associates to every non-empty subset A of X a chosen element $c(A) \in A$.

A choice function is rational if there is an order relation on X so that for every A, $c(A)$ is the maximal element of A according to the order relation.

Rational choice functions correspond to strict order relations. When we allow indifferences we need to enlarge the model to include "choice correspondences:", namely for every nonempty set of alternatives A, $C(A)$ is a non empty subset of A.

Why three models?

Although we can move between these models, they represent different approaches. Preferences refers to what individual desire, choice refers to what individual do, utility refers to the value of matters to individuals. These models are also very different when we try to model irrational behavior.

The notion of utility give us rationality (or transitivity) for free. But the utility function carries more information. $U(a) = 1, U(b) = 2$ and $U(c) = 3$ feels differently than $U(a) = 1, U(b) = 2$ and $U(c) = 100$.

1.3 Voting methods

As a further introduction to social choice theory we will describe several voting methods. We have a set of candidates, each individual has a preference relation on the candidates, and the method gives us a way to choose a candidate.

1. **The plurality method**

The candidate ranked one by most voters is elected.

2. **The Borda count method**

Each voter gives no point to the candidate in the last place, one point to the candidate who is second to last, two points to the candidates third from below and so on. All the points of all the voters are added and the candidate with most points is elected.

3. **Approval voting**

Each voter give one points to candidates he approves and no points to candidates he does not approve. All these points are added and the candidate with most points is elected.

4. **Choosing a Condorcet winner.**

A candidate A is elected if for every other candidate B the majority of voters prefer A to B. (Condorcet's paradox shows that sometimes there is no Condorcet's winner.)

5. **A Multi round method**

We look at the candidate preferred by the least number of voters, delete him from the list of candidates and proceed.

6. **The UW math department method**

We look at those candidates preferred by the least number of voters such that together they are still preferred by less than 50from the list and proceed.

7. **The WI-math department method**
Every member grade the candidates (from 1-100, say). For every candidate we consider his median grade. The candidate with the highest median grade is elected. (Variant: use the ranking of individuals instead of grades.)

8. **An improved approval voting method**

Every voter associates to every candidate either 0 or a polynomial of the form $1+t+t^2+t^3+...+t^k$. These polynomials are added and every candidate have now a polynomial $a_0 + a_1 t + a_2 t^2 + \cdots +$. The winners is determined according to a_0, in case of a tie according to a_1, in case of a tie according to a_3 etc.

9. **Distribute your weight as you wish**

Every voter get 1 unit that he can divide and distribute as he wishes among the candidates. The winner is the candidate who accumulated the most.

Discussion

1) In all these methods there is a complete symmetry between candidates and between voters.

2) All the method we described can lead to a tie and they all can be extended in a straightforward way to describe not only a winner but a social preference relation on the candidates.

3) In most of the methods the outcome depends on the individual preference relations. For the plurality method the outcomes depend on even less - on the individual top candidates. For the approval voting methods as well as methods 7-9 the outcomes depends on some other input from the individuals which does not simply reflect their preference relations.

1.4 Caution about mathematics

We were skeptical at first if mathematics has anything to do with voting methods and collective choice. Convinced otherwise we can move to the other extreme and ask: Can mathematics "solve" this issue and tells us what is the "best" voting method? The short answer is "no".