Zernik

Note Title 6/10/2015

Goal: Any quasi-isom
$F : \mathcal{C} \longrightarrow \mathcal{D}$ admits an inverse
up to homotopy

Notational simplification:

$\mathcal{C}$ has just one object X

$A = \hom_{\mathcal{C}}(X, X)$

$\mu_1 : A \longrightarrow A$

$\mu_2 : A \otimes A \longrightarrow A$

$\mu_3, \dots, \mu_k : A^k \longrightarrow A$

$$0 = \sum \pm \mu_{k_1}(\text{—} \, \mu_{k_2}(\text{—}) \text{—})$$

"A_∞ algebra"

$\mathcal{D}$ also has a single object FX

$\mathcal{B} = \hom_{\mathcal{D}}(FX, FX)$ $\quad \nu_k : \mathcal{B}^{\otimes k} \longrightarrow \mathcal{B}$

F is a functor :

$f_1 : A \longrightarrow B$

$f_2 : A^{\otimes 2} \longrightarrow B$

$\vdots$

$f_k : A^{\otimes k} \longrightarrow B$

$$\sum f_{k_1}(- \mu_{k_2}(-) -) =$$

$$= \sum \nu_k (f_{s_1}(-), \dots, f_{s_k}(-))$$

F is quasi-isomorphism

$H(f_1) : H(A, \mu_1) \longrightarrow H(B, \nu_1)$ is an isom.

Thm. every quasi-isom of A_∞ alg's admit an inverse up to homotopy.

Lemma 1 (Homological Perturbation)

If (A, μ) is an A_∞ algebra, we have:

1) An A_∞ algebra structure $\tilde{\mu}$ on HA w/ $\tilde{\mu}_1 = 0$.

2) A_∞ morphisms

$$\begin{tikzcd} A \arrow[d, shift left, "\pi"] \\ HA \arrow[u, shift left, "I"] \end{tikzcd}$$

3) homotopies $\quad I \circ \pi \simeq \mathbb{1}_A \quad (\pi \circ I \simeq \mathbb{1}_A)$

$\pi_1 \circ I_1 = Id_{HA}$

<u>Lemma 2</u> (Inverse function theorem)

If $f : (A, \mu) \longrightarrow (B, \nu)$ A_∞ morphism, $\mu_1 = 0 = \nu_1$, then f is invertible iff f_1 is invertible.

<u>Pf of Thm</u>:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, shift right, "\pi_A"'] & B \arrow[d, shift left, "\pi_B"] \\ HA \arrow[u, shift right, "I_A"'] \arrow[r, shift right, "H"'] & HB \arrow[l, shift right, "H^{-1}"'] \arrow[u, shift left, "I_B"] \end{tikzcd}$$

$H_1 = \pi_B \circ f_1 \circ I_A = H(f_1)$ invertible

$\Downarrow$ Lemma 2

$\exists$ inverse H^{-1} for H

$g = I_A H^{-1} \pi_B$

$f \circ g = f I_A H^{-1} \pi_B$

$$\simeq I_B \underbrace{(\pi_B f I_A)}_{H} H^{-1} \pi_B$$

$$= I_B \pi_B \simeq \mathbb{1}_B .$$

Similarly $g \circ f \simeq \mathbb{1}_A$ $\blacksquare$

<u>Pf (inverse fn thm)</u>

$A \xrightarrow{f_0} B$ $\qquad g_1 f_1 = \mathbb{1}$

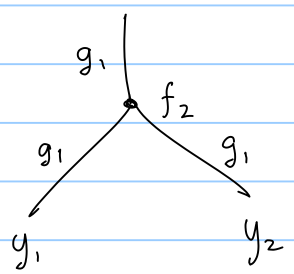
$\forall y_1, y_2 \in B$ $\qquad f_1 g_1 = \mathbb{1}$

Need g_2 s.t.

$$0 = (f \circ g)_2 (y_1, y_2) = f_1 g_2 (y_1, y_2) + f_2 (g y_1, g y_2)$$

$$g_2(y_1, y_2) = - g_1 f_2 (g_1 y_1, g_1 y_2)$$

$g_2 =$



$$0 = f_1 g_{k+1} (y_1, \dots, y_{k+1}) + \sum f_{\geq 2} (g_{\leq k}(-), \dots, g_{\leq k}(-))$$

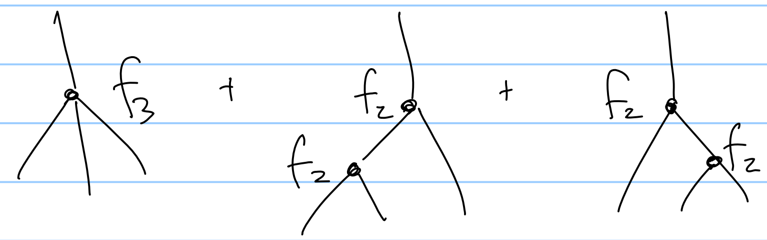
$$\Longrightarrow \quad g_{k+1}(y_1, \dots, y_{k+1}) = \sum g_1 f_{\geq 2} (g_{\leq k}(-), \dots, g_{\leq k}(-))$$

E.g. $g_3(y_1, y_2, y_3) = g_1 f_3(g_1 y_1, g_2 y_2, g_1 y_3) +$

$+ g_1 f_2(g_2(y_1, y_2), y_3) +$

$+ g_1 f_2(y_1, g_2(y_2, y_3))$

$g_3 =$



check that A_∞ morphisms hold for g and that $g \circ f = \mathrm{id}$.

Lemma 1 (HPL)

(A, μ)

Construct linear part

$$HA \underset{\pi_1}{\overset{I_1}{\rightleftarrows}} A \circlearrowleft h$$

$$\mathbb{1}_A - I_1 \pi_1 = dh + hd \qquad d = \mu_1$$

$$\pi_1 h = 0$$

$$h^2 = 0$$

$$h I_1 = 0$$

$$0 \to \operatorname{Im}(d) \xrightarrow{j} \ker(d) \xrightarrow{q} HA \to 0$$
with splittings $\tau_2 : \ker(d) \to \operatorname{Im}(d)$ and $\hat{I}_1 : HA \to \ker(d)$

$$0 \to \ker(d) \xrightarrow{i} A \to \operatorname{Im} d \to 0$$
with splittings $\hat{\pi}_1 : A \to \ker(d)$ and $\tau_1 : \operatorname{Im} d \to A$

$$I_1 = i \hat{I}_1$$

$$\pi_1 = q \hat{\pi}_1$$

$$h = \tau_1 \tau_2 \hat{\pi}_1$$

"$\mu \circ I = I \circ \tilde{\mu}$"

Assume that $\tilde{\mu}_1 = 0, \tilde{\mu}_2, \dots, \tilde{\mu}_k$ have been constructed

I_1 (const'd), $I_2, \dots, I_k$

$$I_{\geq 2} \in \operatorname{Im} h.$$

$$d I_{k+1}(-) + \mu_{\neq 1}(I_{\leq k}(-), \dots, I_{\leq k}(-)) =$$

$$= I_1 \tilde{\mu}_{k+1}(-) + \sum I_{2 \leq \cdot \leq k}(- \tilde{\mu}_{\leq k}(-) -)$$

Apply π_1:

$$\sum \pi_1 \mu_{\neq 1}\left(I_{\le k}(-),\dots,I_{\le k}(-)\right) =$$

$$= \tilde{\mu}_{k+1}(-)$$

Apply h:
$$\begin{pmatrix} hd + dh = I\pi - \mathrm{Id} \\ hd = I\pi - \mathrm{Id} - dh \end{pmatrix}$$

b/c $I_{k+1} \in \mathrm{Im}(h)$

$$\left(\cancel{dh} - \cancel{I_1\pi_1} - \mathbb{1}_A\right) I_{k+1}(-) +$$

$$+ \sum h \mu_{\neq 1}\left(I_{\le k}(-),\dots,I_{\le k}(-)\right) = 0$$

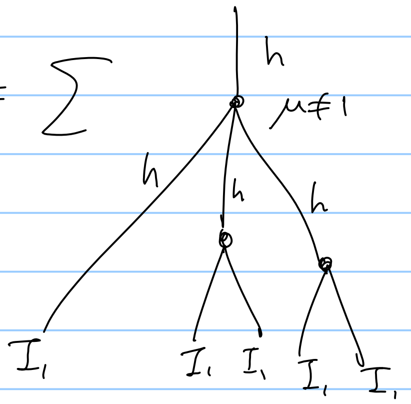
$$\Longrightarrow I_{k+1}(-) = \sum h\mu_{\neq 1}\left(I_{\le k}(-),\dots,I_{\le k}(-)\right)$$

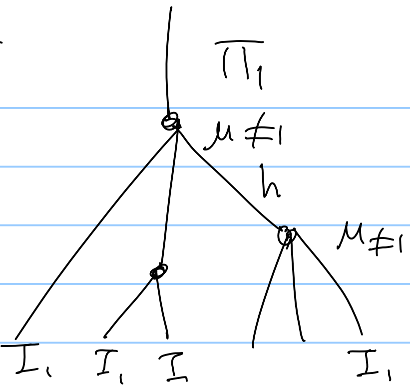
construction of π is similar,

homotopy $I\pi \simeq \mathbb{1}$ $\blacksquare$

Trees:

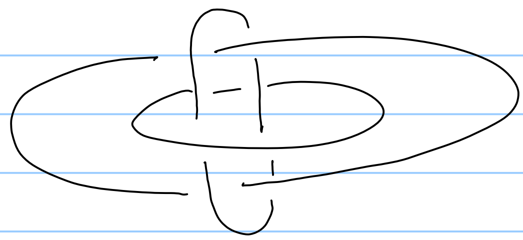
$I = \sum$



$\tilde{\mu} = \sum$ 

Fun Exercise:

$X = S^3 \setminus$ Borromean rings



$$(\underset{\substack{\text{singular}\\ \text{cochains}}}{C^\bullet(X)}, \underset{\mu_1}{\underset{\shortparallel}{\delta}}, \underset{\mu_2}{\underset{\shortparallel}{\cup}}) \qquad \mu_k = 0 \quad k \geq 3$$

Compute A_∞ str on $H^\bullet(X)$

$$H^\bullet(X) = \underbrace{\mathbb{R}\cdot 1}_{0} \oplus \underbrace{\mathbb{R}\langle \gamma_1, \gamma_2, \gamma_3 \rangle}_{1}$$
$$\oplus \underbrace{\mathbb{R}\langle \alpha_{12}, \alpha_{23} \rangle}_{2}$$

Show

$\tilde{\mu}_1 = 0$ (by constr'n)

$\tilde{\mu}_2 = 0$ (no two rings linked)

$$\tilde{\mu}_3(\gamma_1, \gamma_2, \gamma_3) = \alpha_{12} + \alpha_{23}$$

(non-zero reflects fact rings are linked)