Recall:

We constructed $L : S^{n-2} \hookrightarrow \mathbb{CP}^{n-2} \setminus D$

We computed $\mathcal{A} := CF^{*}_{\mathcal{F}(\mathbb{CP}^{n-2}, D^{\underline{n}})}(L, L)$:

$\mathcal{A} \cong A \otimes R$ as R-module,

$R := \mathbb{C}[\![r_1, \dots, r_n]\!]$

$A := \mathbb{C}[\theta_1, \dots, \theta_n]$

both graded in $H_1(\mathcal{P}^{n-2}) \cong \mathbb{Z} \oplus \mathbb{Z}^n / 2(1-n) \oplus (1, \dots, 1)$

The A_∞ structure $\mu^{\bullet}$ is characterised (up to an automorphism of R and A_∞ quasi-iso.) by

- μ_0^2 = wedge product
- $\mathrm{Sym}(n)$-equivariance
- $\Phi_{HKR}(\mu^{\geq 3}) \cong \pm z_1 \dots z_n \pm \sum_i r_i z_i^n + \mathcal{O}(r^2)$.

Now we look for a similar structure on the B-model.

We consider

$$S := R[z_1, \dots, z_n]$$

$$W \in S, \; W := z_1 \dots z_n + \sum_i r_i z_i^n .$$

equipped with a grading in $H_1(\mathcal{P}^{n-2})$ as above: W has $\deg = 2$ so we can introduce the R-linear DG category of $H_1(\mathcal{P}^{n-2})$-graded matrix factorizations of W.

Further, we have the MF corresponding to the skyscraper, $\mathcal{O}_0$. By the previous talk,

DG alg. $\qquad\qquad\qquad\qquad$ A_∞ alg.

$$\hom^*_{MF^G(S,W)}(\mathcal{O}_0, \mathcal{O}_0) \xrightarrow{\sim} (A \otimes R, \mu^\bullet)$$

$H_1(\mathcal{P}^{n-2})$-graded R-linear A_∞ quasi-iso.

Further,

① μ^2 = wedge product

② the construction can be done $\mathrm{Sym}(n)$-equivariantly

③ The minimal model satisfies

$$\Phi(\mu^{\geq 3}) = \pm z_1 \ldots z_n \pm \sum r_i z_i^n .$$

<u>Cor</u>: Let $\mathcal{A} := \hom^\bullet_{\mathcal{F}(\mathbb{CP}^{n-2}, D^{\underline{n}})}(L, L)$

$\mathcal{B} := \hom^\bullet_{MF^G(S,W)}(\mathcal{O}_0, \mathcal{O}_0)$.

Then there is an automorphism

$$\psi : R \xrightarrow{\sim} R$$
$$r_i \longmapsto r_i \cdot \varphi(T)$$

where $\varphi(T) = \pm 1 + c_1 T + \ldots$

and an R-linear A_∞ quasi-iso

$$\mathcal{A} \xrightarrow{\sim} \psi^* \mathcal{B}.$$

Now we got our hands on some mirror symmetry: $(\mathbb{CP}^{n-2}, D^{\underline{n}})$ is mirror to $(\mathrm{Spec}\, S, W)$.

Recall (talk 1): There is an embedding

$$p^* \mathcal{F}(\mathbb{CP}^{n-2}, D^n) \hookrightarrow \mathcal{F}(X^n, D),$$

where $p : H_1(\mathcal{G}(X^n \setminus D)) \longrightarrow H_1(P^{n-2})$ is the map induced by the cover. In fact

$$\begin{tikzcd}
H_1(\mathcal{G}(X^n \setminus D)) \cong \mathbb{Z} \oplus H_1(X^n \setminus D) \arrow[d, "q"] \\
\mathbb{Z}
\end{tikzcd}$$

So we can define a $\mathbb{Z}$-graded relative Fukaya category, $q_* \mathcal{F}(X^n, D)$, which forgets the $H_1(X^n \setminus D)$ part of the grading.

So we have

$$\begin{tikzcd}
q_* p^* \underline{A} \arrow[d, phantom, sloped, "\cong"] & \subset q_* \mathcal{F}(X^n, D) & \\
q_* p^* \psi^* \underline{B} & \subset q_* p^* MF^{G}(S, W) \arrow[r, leftrightsquigarrow, "?"'] & D^b Coh(Y^n).
\end{tikzcd}$$

add all shifts = look at all lifts of L

now we explain this.

Defn: Let S be a $\mathbb{Z}$-graded ring, $W \in S$ of pure degree n. Consider the grading datum $G_{MF(n)}$

$$\mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z}/_{2\oplus -n}$$

$$j \longmapsto j \oplus 0$$

We equip S with a $G_{MF(n)}$-grading by putting S_j in degree $0 \oplus j$; so W has degree $2 \in G_{MF(n)}$.

We then define the $\mathbb{Z}$-graded DG category

$$Gr\,MF(S, W) := f^* MF^{G_{MF(n)}}(S, W).$$

__Thm__ (Orlov): If $S = \underset{\substack{\uparrow \\ \text{field}}}{\mathbb{K}}[z_1, \dots, z_n]$, $W \in S$ pure of degree n, and the hypersurface

$Y := \{W = 0\} \subset \mathbb{P}^{n-1}_{\mathbb{K}}$

is smooth, then there is an equivalence of triangulated categories,

$$D^b\mathrm{Coh}(Y) \cong H^0(\mathrm{GrMF}(S, W)).$$

This lifts to a quasi-equivalence of the underlying DG categories, by general results of Lunts and Orlov.

How to compare this with $p^* MF^G(S, W)$?

__Prop__: $H_1(\mathcal{G}\mathcal{P}^{n-2}) \cong \mathbb{Z} \oplus \mathbb{Z}^n / 2(1-n) \oplus (1, \dots, 1)$

$$H_1(\mathcal{G}(X^n \setminus D)) \cong \underset{\substack{\uparrow \\ \mathcal{G}(X^n)\ \text{trivial bundle}}}{\mathbb{Z}} \oplus \underset{\substack{\nwarrow \\ \qquad H_1(X^n \setminus D).}}{\mathbb{Z}^n/(1, \dots, 1)}$$

The map p sends

$$j \oplus (j_1, \dots, j_n) \mapsto j + 2(1-n)(j_1 + \dots + j_n) \oplus (nj_1, \dots, nj_n).$$

There is a commutative square of grading data,

$$\begin{tikzcd}
j \oplus (j_1, \dots, j_n) \arrow[r, mapsto] & j \\
H_1(\mathcal{G}(X^n \setminus D)) \arrow[r, "q"'] \arrow[d, "p"] & \mathbb{Z} \arrow[d, "f"] \\
H_1(\mathcal{G}\mathcal{P}^{n-2}) \arrow[r, "g"] & G_{MF(n)} \\
j \oplus (j_1, \dots, j_n) \arrow[r, mapsto] & j \oplus (n-1)(j_1 + \dots + j_n]
\end{tikzcd}$$

(Check:

$$\begin{tikzcd}
0 \oplus e_i \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
2(1-n) \oplus ne_i \arrow[r, mapsto] & 2(1-n) \oplus (n-1)n = 0
\end{tikzcd}$$

).

<u>Note</u>: If V is $H_1(\mathcal{G}\mathbb{P}^{n-2})$-graded,

f^*g_*V admits a Γ-grading,

$$\Gamma := H_1(\mathcal{G}\mathbb{P}^{n-2}) / H_1(\mathcal{G}(X^n \setminus D)) \cong (\mathbb{Z}/n)^n / (1, \ldots, 1).$$

Equivalently, it has a Γ^*-action,

$$\Gamma^* := \mathrm{Hom}(\Gamma, \mathbb{C}^*).$$

<u>Ex</u>: $\underset{\substack{\| \\ (f^*g_*V)_{0\in\Gamma}}}{(f^*g_*V)^{\Gamma^*}} \cong q_*p^*V.$

<u>Cor</u>:

$$\begin{array}{ccccc}
q_*\mathcal{F}(X^n, D) & & & & \mathrm{GrMF}^{\Gamma^*}(S, W) \\
\cup & & & & \cup \\
q_*p^*\underline{\mathcal{A}} & \cong & q_*p^*\psi^*\underline{\mathcal{B}} & \cong & (f^*g_*\psi^*\underline{\mathcal{B}})^{\Gamma^*}
\end{array}$$

Here, S has a Γ^*-action, objects of GrMF^{Γ^*} are Γ^*-equivariant MFs, homs are Γ^*-equiv't part of $\hom_{\mathrm{GrMF}}$.

Orlov's theorem $\Rightarrow$ over a <u>field</u>,

$$\mathrm{GrMF}^{\Gamma^*}(S, W) \cong D^b\mathrm{Coh}^{\Gamma^*}(\{W = 0\}).$$

In particular we can tensor with

$$\mathbb{K}_A := \mathbb{C}((r)) \qquad \mathbb{K}_B := \mathbb{C}((q))$$

$$\begin{tikzcd}
 & \Psi^* D^b \mathrm{Coh}^{\Gamma^*}(\{W=0\} \subset \mathbb{P}^{n-1}_{\mathbb{K}_B}). \arrow[d, phantom, "\wr\|"] \\
q_* \mathcal{F}(X^n, D) \otimes_R \mathbb{K}_A & \Psi^* \mathrm{Gr\,MF}^{\Gamma^*}(S \otimes \mathbb{K}_B, W \otimes 1) \\
\cup & \cup \\
(q_* p^* \underline{\mathcal{A}}) \otimes_R \mathbb{K}_A \arrow[r, phantom, "\cong"] & \Psi^*(f^* g_* \underline{\mathcal{B}} \otimes \mathbb{K}_B)^{\Gamma^*}
\end{tikzcd}$$

↑ this subcategory split-generates (next)

↑ this subcategory split-generates (next talk)

$$\Rightarrow \boxed{D^\pi \mathcal{F}(X^n, D) \cong \Psi^* D^b \mathrm{Coh}^{\Gamma^*}(Y^n)}$$

↑ $\mathbb{Z}$-graded, $\mathbb{K}_A$-lin. version.

↑ replace 'π' by 'b': $D^b\mathrm{Coh}$ is split-closed.

⇒ Theorem is proved, modulo split-generation!