Deformation theory

Recall: we constructed

$$L : S^{n-2} \hookrightarrow \mathbb{CP}^{n-2} \setminus D =: \wp^{n-2}$$

Set $\mathcal{A}^k := \hom^*_{\mathcal{F}(\mathbb{CP}^{n-2}, D^k)}(L, L)$

an A_∞ algebra over $R = \mathbb{C}[\![r_1, \dots, r_n]\!]$,

$\mathcal{A}^k \cong A \otimes R$ as a vector space,

where $A := \mathbb{C}[\theta_1, \dots, \theta_n]$.

It is graded in
$H_1(\mathcal{L}\wp^{n-2}) \cong \mathbb{Z} \oplus \mathbb{Z}^n / 2(1-n) \oplus (1, \dots, 1)$.

We compute:

$$\deg \theta_i = -1 \oplus (0, \dots, \underset{i}{1}, \dots, 0)$$

$$\deg r_i = 2 - 2k \oplus (0, \dots, k, \dots, 0)$$

Note: $\underset{\uparrow \deg\text{-}0}{R_0} \cong \mathbb{C}[\![T]\!]$, $T := r_1 \dots r_n$.

We wish to determine $\mathcal{A}^n$, up to A_∞ quasi-iso, as that determines the A_∞ struc. on lifts of L to X^n (lecture 1).

Recall the HKR map

$$\Phi : CC^\bullet(A \otimes R) \longrightarrow R[\![z_1, \dots, z_n]\!][\theta_1, \dots, \theta_n]$$

$$\varphi \longmapsto \varphi(\underline{z}, \dots, \underline{z})$$

where $\underline{z} := z_1\theta_1 + \dots + z_n\theta_n$.

$$\deg z_i = 2 \oplus (0, \dots, -1, \dots, 0)$$

Thm: If $\mu^\bullet$ is an R-linear A_∞ structure on $A \otimes R$,

$\mu^\bullet = \mu^\bullet_0 + \mu^\bullet_1 + \ldots$ its expansion in powers of r_i, and

① It is $H_1(\mathcal{G}\mathcal{P}^{n-2})$-graded

② It is $\mathrm{Sym}(n)$-equivariant

② μ^2_0 = exterior product

③ $\Phi(\mu^{\geq 3}) = \pm z_1 \ldots z_n \pm \sum_i r_i z_i^n + \mathcal{O}(r^2)$

and if $m^\bullet$ is another such, then there exist

ⓐ $\varphi(T) = \pm 1 + c_1 T + \ldots \in \mathbb{C}[\![T]\!]$

ⓑ An R-linear A_∞ homomorphism

$$F : (A \otimes R, \mu^\bullet) \to \psi^*(A \otimes R, m^\bullet)$$

where $\psi : R \xrightarrow{\sim} R$
$\psi(r_i) = r_i \cdot \varphi(T)$

and
$F^1_0 = \mathrm{id}$.

(note: we will prove $\mathcal{A}^n$ has these properties presently).

Proof: We work order-by-order in the r_i.

First, we determine the $\mathbb{C}$-linear A_∞ alg. $\mathcal{A}_0 := (A \otimes R, \mu^\bullet_0)$.

Grading $\Rightarrow \mu^s = 0$ unless $s \equiv 2 \pmod{n-2}$

Hence, $\mu^s = 0$ for $2 < s < n$. It follows that

$$[\mu^2, \mu^n] = 0 \Rightarrow [\mu^n] \text{ defines a class in } HH^2(A)^{2-n}.$$

<u>Prop</u>: If $\mu_0^{\geq 2}$ and $m_0^{\geq 2}$ are two A_∞ structures on A, $\mu_0^2 = m_0^2 =$ product on A,

$\mu_0^s = m_0^s = 0$ for $2 < s < n$,

$[\mu_0^s] = [m_0^s]$ in $HH^2(A)^{2-n}$,

and $HH^2(A)^s = \begin{cases} \mathbb{C} & \text{for } s = n \\ 0 & \text{else} \end{cases}$

then there is an A_∞ q.iso.

$F_0 : (A, \mu_0^\bullet) \to (A, m_0^\bullet)$.

In our case, HKR$\Rightarrow$

$$HH^*(A) \cong \mathbb{C}[[z_1, \dots, z_n]][\theta_1, \dots, \theta_n]$$

<u>Lem</u>: $HH^2(A)^s = \begin{cases} \mathbb{C} \cdot z_1 \dots z_n & \text{if } s = n \\ 0 & \text{otherwise} \end{cases}$

and $[\mu_0^s] = [m_0^s] = \pm z_1 \dots z_n$

$\Rightarrow (A, \mu_0^\bullet) \cong (A, m_0^\bullet)$ A_∞ quasi-iso.

Now we need to study the higher-order terms in the r_i.

If $\mu^\bullet = \mu_0^\bullet + \sum r_i \mu_{1,i}^\bullet + \ldots$

then $[\mu_0^\bullet, \mu_i^\bullet] = 0 \quad \forall i$

$\Rightarrow$ they define classes in $HH^\bullet(\mathcal{A}_0)$, called <u>first-order deformation classes.</u>

They live in $(HH^\bullet(\mathcal{A}_0) \otimes \mathfrak{m}/\mathfrak{m}^2)^2$.

<u>Defn</u>: The <u>length filtration</u> on $CC^\bullet(\mathcal{A}_0)$:

$$F^{\geq s} CC^\bullet(\mathcal{A}_0) := \prod_{p \geq s} \mathrm{Hom}(\mathcal{A}_0^{\otimes p}, \mathcal{A}_0).$$

<u>Defn</u>: Truncated Hochschild cohomology is

$$\mathrm{Tr}HH^\bullet(\mathcal{A}_0) := H^\bullet(F^{\geq 1} CC^\bullet(\mathcal{A}_0)).$$

<u>Prop</u>: If $\mu^\bullet$ and $m^\bullet$ are two minimal A_∞ structures on $A \otimes_{\mathbb{C}} R$, with

$$\mu^\bullet = \mu_0^\bullet + \sum r_i \mu_i^\bullet + \ldots$$

$$m^\bullet = m_0^\bullet + \sum r_i m_i^\bullet + \cdots,$$

$$(A, \mu_0^\bullet) = (A, m_0^\bullet) = \mathcal{A}_0,$$

$[\mu_i^\bullet] = [m_i^\bullet] \in \mathrm{Tr}\,HH^\bullet(\mathcal{A}_0)$, and the classes $[\mu_i^\bullet] \otimes r_i$ span $(\mathrm{Tr}\,HH^\bullet(\mathcal{A}_0) \otimes R)^2$ as an R_0-module ($R_0 = \mathbb{C}[[T]]$), Then there exist

1) An automorphism

$$\Psi : R \longrightarrow R$$
$$\Psi(r_i) = r_i \cdot (1 + c_1 T + c_2 T^2 + \dots)$$

2) An A_∞ homomorphism

$$F : (A \otimes R, \mu^\bullet) \rightarrow (A \otimes R, m^\bullet)$$

with $F_0 = id$.

Proof: Construct Ψ and F, order-by-order in r_i. If

$$F = F_0 + F_1 + F_2 + \dots$$

$\Psi(r_i) = r_i (1 + \Psi_1 + \dots)$ are such that

$$\star := \sum F(\dots \mu^\bullet(\dots)\dots) - \sum m^\bullet (F^\bullet(\dots) \dots F^\bullet(\dots)) = 0$$

to order N-1 in the r_i, we choose

$F_N \in \left(\mathrm{Tr}\, HH^\bullet(\mathcal{A}_0) \otimes R_N\right)^1$,

$\Psi_N(r_i)$

so that $\star$ holds to order N.

Point: $\star_N$ is closed, so defines a class in $\left(\mathrm{Tr} HH^\bullet(\mathcal{A}_0) \otimes R_N\right)^2$;

this class is

$$[\star_N] = \sum_i c_i [\mu_i^\bullet];$$

so if we set

$$\psi_N(r_i) = c_i \cdot r_i$$

then $[\star_N] = 0$ in $(\mathrm{Tr}\,HH^\bullet, \ldots)^2$

$\Rightarrow$ we can choose F_N s.t.

$$\partial F_N = \star_N$$

then $\star_N \simeq 0$.

<u>Lem</u>: The class $\sum_i [r_i \mu_i^\bullet]$ spans the $\mathrm{Sym}(n)$-equivariant part of $(\mathrm{Tr}\,HH^\bullet(\mathcal{A}_0) \otimes R_0)^2$.

<u>Pf</u>: We have a spectral sequence

$$HH^\bullet(A) = \mathbb{C}[\![z_i]\!](\theta_i) \Rightarrow HH^\bullet(\mathcal{A}_0)$$

induced by the length filtration.

Compute: the class $\sum_i r_i z_i^n$ spans

$$(R[\![z_i]\!][\theta_i]^2)^{\mathrm{Sym}(n)} \cong (HH^\bullet(A \otimes R)^2)^{\mathrm{Sym}(n)}$$

as R_0-module; $\mu_i^\bullet = z_i^n$ to leading order in length filtration, so the result follows.

Applying the above proposition in a $\mathrm{Sym}(n)$-equivariant context, the Theorem follows.

Prop: $CF^*_{\mathcal{F}(\mathbb{CP}^{n-1}, D^n)}(L,L) =: \mathcal{A}^n$ has the properties listed in the theorem.

Pf: From last lecture,

$$\mathcal{A}^n_0 = \mathcal{A}^1_0$$

has μ^2_0 as required, and

$$\Phi(\mu_0^{\geq 3}) = \pm z_1 \cdots z_n$$

as required; we need to check that

$$\Phi(\mu_1^{\geq 3}) = \pm \sum_i r_i z_i^n .$$ 1st-order deformation class

We computed that the 1st-order def. class for $\mathcal{A}^1$ was

$$\pm \sum_i r_i z_i .$$

We apply:

Lem: If the 1st-order def class of $\mathcal{F}(X,D)$ is

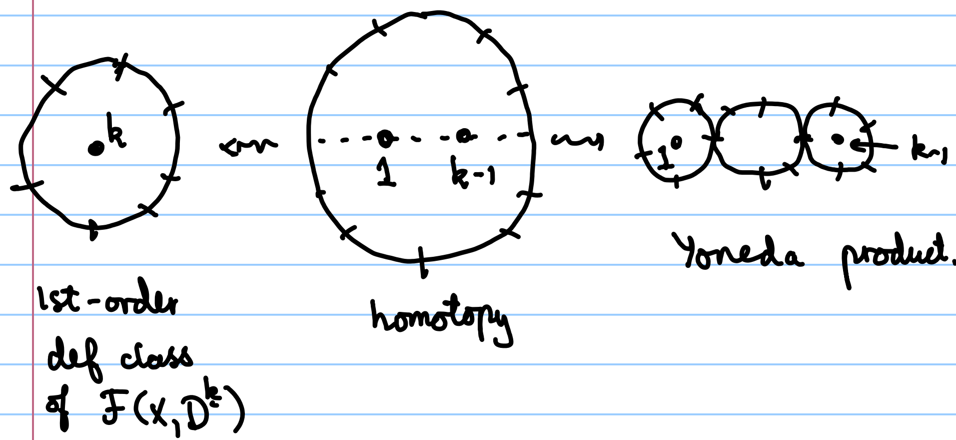
$$\sum_i r_i \alpha_i$$

then the 1st-order def class of

$\mathcal{F}(X, D^{\underline{k}})$ is

$$\sum_i r_i \alpha_i^k$$ power denotes Yoneda product on HH^*.

Pf: By induction on k. We consider moduli spaces



(cf. the proof that CO is an algebra hom.).

<u>Cor</u>: The 1st-order def class of $\mathcal{A}^n$ is

$$\sum r_i z_i^n.$$

Finally, $\mathcal{A}^n$ is Sym(n)-equivariant: actually it's not, but it's quasi-isomorphic to such by Seidel's trick.