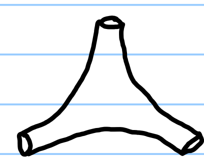
Talk 2

So, we want to understand $\mathcal{F}(\mathbb{CP}^{n-2}, D^{\underline{n}})$.

The first step is to understand $\mathcal{F}(\mathbb{CP}^{n-2} \setminus D)$.

$\mathbb{CP}^{n-2} \setminus \underset{n \text{ hyperplanes}}{\underset{\shortparallel}{D}}$ is called the 'pair of pants'

When $n=3$, it is



(and in higher dim's, Mikhalkin explains how to use tropical geometry to decompose hypersurfaces into pairs of pants).

<u>Prop</u>: There exists an exact Lagrangian immersion

$$L : S^{n-2} \looparrowright \mathbb{CP}^{n-2} \setminus D$$

so that

⓪ This is defined!

① $CF^*(L,L) \cong HF^*(L,L)$ (i.e. $\mu^1 = 0$)

$$\cong \Lambda^{\cdot} \mathbb{C}^n \quad \text{(as an algebra)}$$

$$\cong \mathbb{C}[\theta_1, \dots, \theta_n]$$

(in $\mathcal{F}(\mathbb{CP}^{n-2} \setminus D)$. L^1 has <u>non-triv.</u> spin str.!)

② $\mu^n(\underline{z}, \dots, \underline{z}) = \pm z_1 \dots z_n$

dummy variables

where $\underline{z} = z_1 \theta_1 + \dots + z_n \theta_n$

(still in $\mathcal{F}(\mathbb{CP}^{n-2} \setminus D)$)

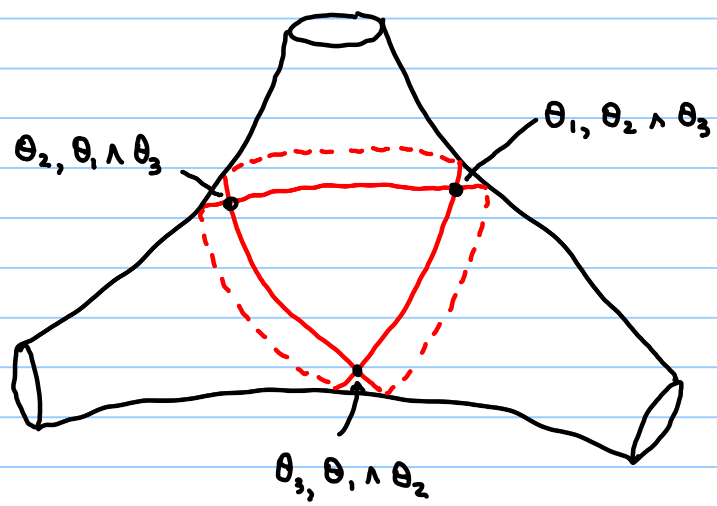
③ In $\mathcal{F}(\mathbb{CP}^{n-2}, D)$, we have

$$\mu^1(\theta_i) = \pm r_i \cdot 1 + O(r^2).$$

(N.B. Implicitly determines grading)

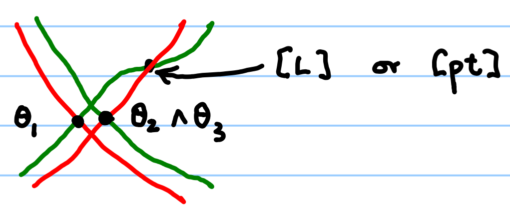
[Do this by counting discs... it will turn out these determine everything else, as we will see next time].

E.g. When $n=3$, L looks like:



$$CF^*(L,L) = \mathbb{C}\langle [L], [pt], \theta_1, \theta_2, \theta_3, \theta_1 \wedge \theta_2, \theta_1 \wedge \theta_3, \theta_2 \wedge \theta_3 \rangle$$

[explain why get two generators per self-intersection:



]

Check: • $\mu^1 = 0$

• $\mu^2(\theta_1, \theta_2) = \theta_1 \wedge \theta_2$

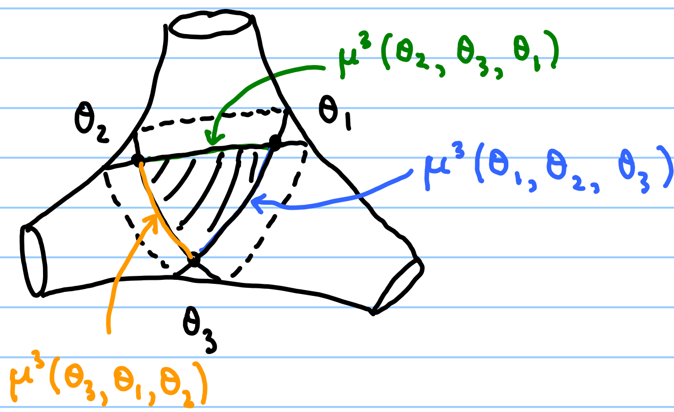
$\mu^2(\theta_2, \theta_1) = -\theta_1 \wedge \theta_2$ } triangles

• $\mu^2(\theta_1, \theta_2 \wedge \theta_3) = [pt]$ (Poincaré duality)

• $\mu^2([L], x) = x$ (unitality)

$\Rightarrow$ (1)

• $\mu^3(\underline{z}, \underline{z}, \underline{z}) = \pm z_1 z_2 z_3$



$\Rightarrow$ (2)

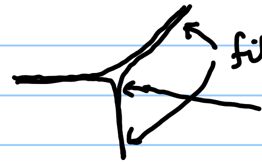
• $\mu^1(\theta_i) = \pm r_i \cdot 1$ in $\mathcal{F}(\mathbb{CP}^1, D)$:

Now, how do things work in higher dimensions?

$$\begin{tikzcd}[column sep=small]
\mathbb{CP}^{n-2} \setminus D \arrow[d, phantom, "\shortparallel" rotate=90] & = \{\Sigma z_i = 0\} \subset & \mathbb{CP}^{n-1} \setminus \cup\{z_i = 0\} \arrow[d, phantom, "\cong" rotate=90] & \\
\mathcal{P}^{n-2} & & (\mathbb{C}^*)^{n-1} \arrow[dl, "\mathrm{Log}"'] \arrow[dr, "\mathrm{Arg}"] & \\
& \mathbb{R}^{n-1} & & (S^1)^{n-1}
\end{tikzcd}$$

$\mathrm{Log}(\mathcal{P}^{n-2})$ = "amoeba"

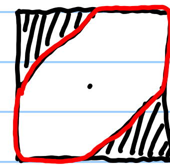
$n=3$



fibres = S^1

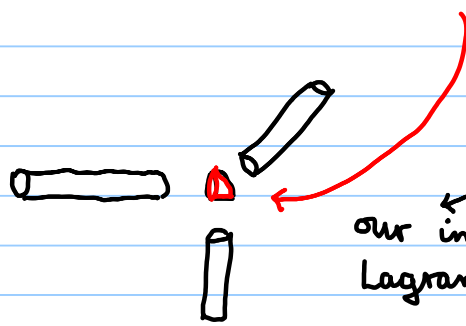
fibre = coamoeba C_1

$\mathrm{Arg}(\mathcal{P}^{n-2})$ = "coamoeba"



C_1

Picture:



hexagon $= \partial C_1$

our immersed Lagrangian

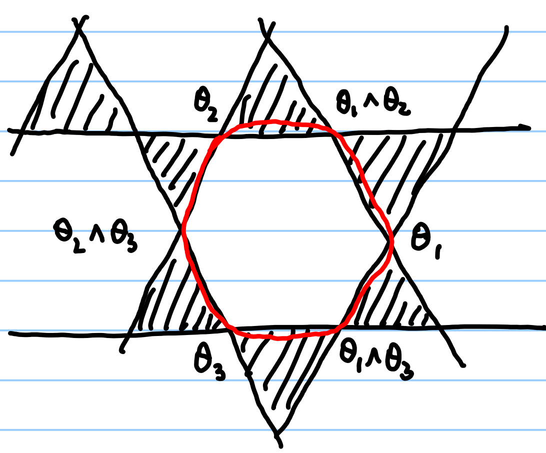
Note: $(\theta_1, \theta_2, \theta_3) \in (S^1)^3/S^1$ lie in $\mathrm{Arg}(\mathcal{P}^1)$

only if $\sum r_j e^{i\theta_j} = 0$ for some j

$\Leftrightarrow$ $e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}$ are not contained in a halfspace

$\Rightarrow \overline{\mathrm{Arg}(\mathcal{P}^1)}^c = \mathrm{im}\left([0,\pi)^3\right) \subset (S^1)^3/S^1$.

A more symmetric picture of univ. cover:



self-intersections

$\updownarrow$

vertices

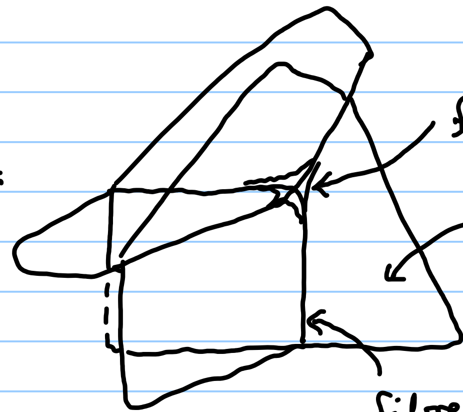
$\shortparallel$

$(\pi, 0, 0)$	θ_1
$(0, \pi, 0)$	θ_2
$(0, 0, \pi)$	θ_3
$(\pi, \pi, 0)$	$\theta_1 \wedge \theta_2$

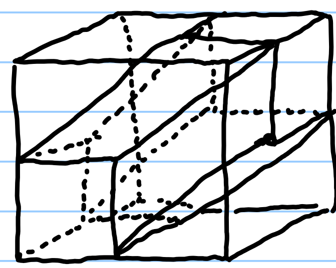
...

$n = 4$:

$\mathrm{Log}(\mathcal{P}^2) =$



Again, $C_2 = \mathrm{im}\left([0,\pi)^4\right) \subset (S^1)^4/S^1$



"rhombic dodecahedron"

Self-intersections $=$ vertices

$= \pi \cdot (\mathbb{Z}_2)^4 \setminus \{(0,0,0,0), (\pi,\pi,\pi,\pi)\}$

$\longleftrightarrow$ generators of $\mathbb{C}[\theta_1, \theta_2, \theta_3, \theta_4]$
<u>except</u> 1 and $\theta_1 \wedge \ldots \wedge \theta_4$:
those correspond to $H^*(S^2)$.

<u>Lem</u>: We have a Lagrangian immersion

$$L : S^{n-2} \looparrowright \mathcal{P}^{n-2}$$

so that $\mathrm{Arg} \circ L$ approximates ∂C^{n-2}, and self-intersections are in 1-1 correspondence with vertices.

<u>Construction</u>: Take a Weinstein nbhd for $\mathbb{RP}^{n-2} \subset \mathbb{CP}^{n-2}$. Construct L as

$$
\begin{tikzcd}
S^{n-2} = \Gamma(\varepsilon df) \subset D^*_\varepsilon S^{n-2} \arrow[d, "2:1"] \arrow[dd, bend right, "L"'] \\
D^*_\varepsilon \mathbb{RP}^{n-2} \arrow[d, hook] \\
\mathbb{CP}^{n-2}
\end{tikzcd}
$$

Problem: L must avoid D. Note D intersects $\mathbb{RP}^{n-2}$ in hyperplanes $D_i = \{z_i = 0\}$, but as long as ∇f is transverse to these hyperplanes, pushing off by εdf will push L off of D.

[if we make the embedding

$$D^*_\varepsilon \mathbb{RP}^{n-2} \hookrightarrow \mathbb{CP}^{n-2}$$

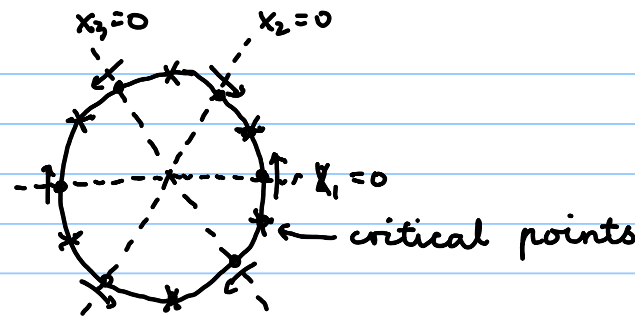
be J-holomorphic along $\mathbb{RP}^{n-2}$, i.e. take $\partial_t(p, t\theta) := J(g^{-1}\theta)$ as J on $D^*\mathbb{RP}^{n-2}$].

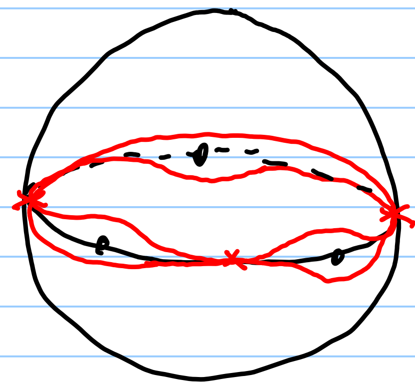
$$
\begin{array}{ccl}
S^{n-2} & = & \{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum x_i = 0, \sum x_i^2 = 1\} \\
\downarrow{\scriptstyle 2:1} & & \quad\quad \downarrow \\
\mathbb{RP}^{n-2} & & \quad [x_1 : \dots : x_n].
\end{array}
$$

We take $f := \sum_j g(x_j)$ where:

So we can draw ∇f:

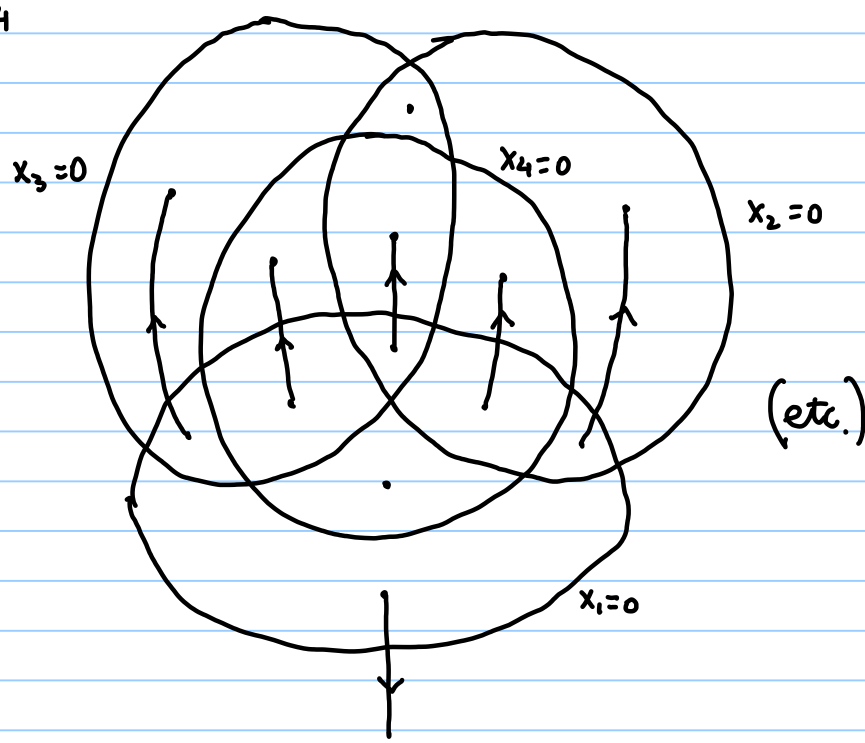
$n=3$





Note g is odd $\Rightarrow$ so is f, so L has self-intersections precisely at the critical points of f (otherwise the two sheets of S^{n-2} get pushed in opposite directions).

$n=4$



The hyperplanes split S^{n-2} into regions indexed by subsets

$$K \subset \{1, \dots, n\},$$

namely $S_K := \{ x_i > 0 \text{ for } i \in K, \; x_i < 0 \text{ for } i \notin K \}$.

All subsets except $K = \phi, \{1, \dots, n\}$ are realized.

f is Morse, with one critical point p_K in each region: So

$$CF^*(L, L) \cong C^*(S^{n-2}) \oplus \bigoplus_{\substack{K \subset [n] \\ K \neq \phi, [n]}} \mathbb{C} \cdot p_K$$

$$\cong \mathbb{C}[\theta_1, \dots, \theta_n] \quad \text{(as vec. space).}$$

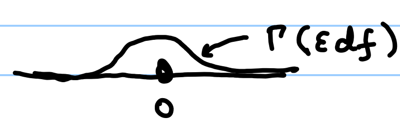
($\circledast$: L can be lifted to a cover, where it's <u>embedded</u>)

Also note: Arg sends critical point p_K to vertex

$$\pi \cdot \sum_{i \notin K} e_i \in \left(\mathbb{R}^n / 2\pi \mathbb{Z}^n \right) \Big/ \left(\mathbb{R} / 2\pi \mathbb{Z} \right)$$

e_i: ith coord vector

of C^{n-2}. In fact it crushes everything in S_K to this point, but as we cross a hypersurface, one variable x_i changes from $\mathrm{Arg} = \pi$ to $\mathrm{Arg} = 0$ via $\mathrm{Arg} \in (0, \pi)$:



So Arg maps the hyperplane arrangement to C^{n-2}, realizing a duality with the polytope.

Note: you can see what the H_1-degrees of our generators θ_i are from the coamoeba picture. Namely,

$$H_1(\mathcal{P}^{n-2}) \cong \mathbb{Z}^n/\mathbb{Z}$$

and p_K has degree $\sum_{i \notin K} e_i$.

So there are no possible differentials between these classes:

$$CF^*(L,L) \cong HF^*(L,L) \cong \mathbb{C}[\theta_1, \ldots, \theta_n]$$

And the only possible products have the form

$$\mu^2(p_K, p_J) = p_L$$

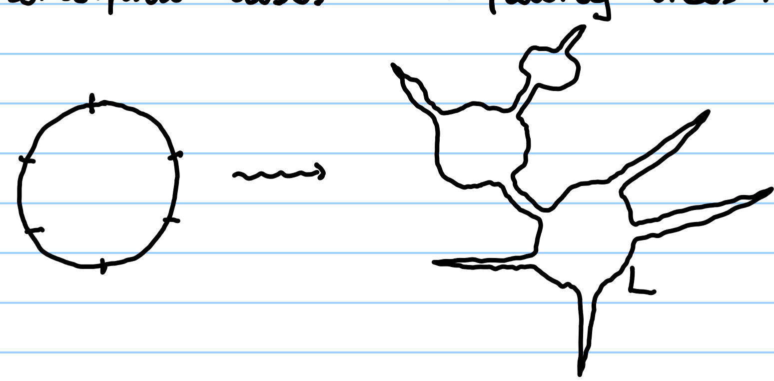
where either
- $L = K \sqcup J$
- $K \cup J = \{1, \ldots, n\}$ and $L = K \cap J$.

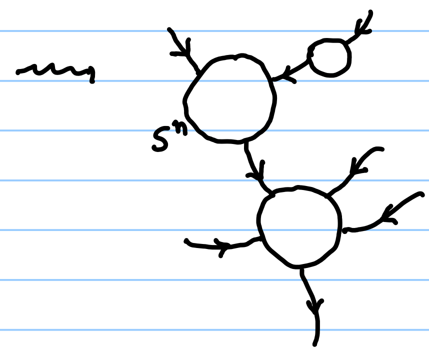
(if we consider full $\mathbb{Z} \to H_1(gX)$ grading, only first one can happen).

How to compute that $\mu^2(p_J, p_K) = \pm p_{J \sqcup K}$?

Consider the limit as $L \rightsquigarrow$ double cover of $\mathbb{RP}^{n-2}$

Holomorphic discs $\rightsquigarrow$ 'nearly trees':





$$CF^*_{\text{pearly}}(L,L) = CM^*(f) \oplus CM^*(h)$$

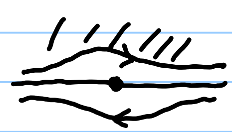
$\uparrow$ Morse function on S^n

$$\cong \mathbb{C}[\theta_1,\dots,\theta_n].$$

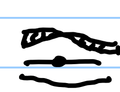
A_∞ operations defined by counting 'pearly trees': 'boundary' comes with a lift to S^n; flowlines are flowlines of f (if opposite sides of the edge are labelled by opposite sheets of the double cover) or h (if opposite sides are labelled by the same sheet).

One can define 'intersection number' of a pearly tree with D: if ∂(disc) crosses $D_{\mathbb{R}}$ in <u>positive</u> direction: $+0$

<u>neg</u> direction: $+1$



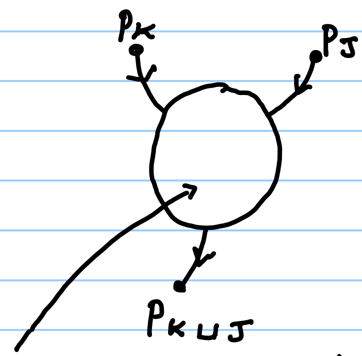
if an f-flowline crosses $D_{\mathbb{R}}$: $+1$ 

" " h-flowline " " : $+0$ 

internal intersections with D: $+1$. 

To start with, we only care about $\mathcal{F}(S^{n-2})$, so count pearly trees with int. # = 0.

Let's determine the coefficient of $p_{J\sqcup K}$ in $\mu^2(p_J, p_K)$. Int. # $= 0$ $\Rightarrow$ total degree of curve is 1 $\Rightarrow$ we have half a real 'line' in $\mathbb{CP}^n$:



line hitting ascending mflds of p_K, p_J, descending mfld of $p_{K\sqcup J}$.

Let $F_{K_1,\dots,K_k} := \{\underline{x} \in S^n : x_i = x_j \ \forall\, i,j \in K_\ell, \forall\, \ell\}$

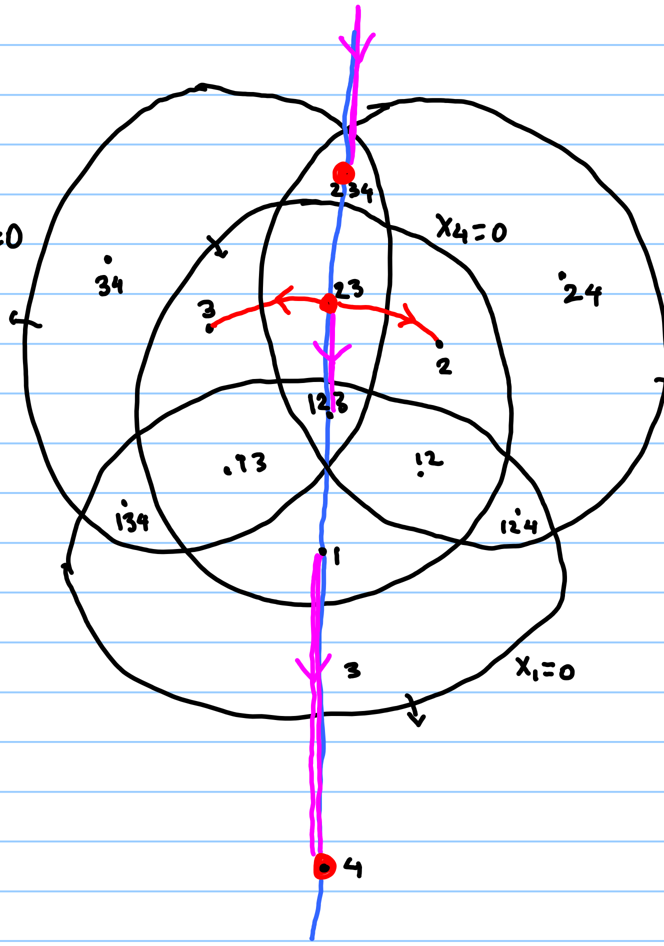
Then by symmetry, asc. mfld of p_K is an open subset of $F_{\bar{K}}$, desc. mfld is an open subset of F_K.

If a line hits the subspaces $F_{\bar{K}_1}$, $F_{\bar{K}_2}$, it is contained in $F_{\bar{J}\cap\bar{K}} = F_{\overline{J\sqcup K}}$

This intersects $F_{J\sqcup K}$ in a unique point $p_{J\sqcup K}$: so the unique line hitting $F_{\bar{J}}$, $F_{\bar{K}}$, $F_{J\sqcup K}$ is $F_{J,K,\overline{J\sqcup K}}$,

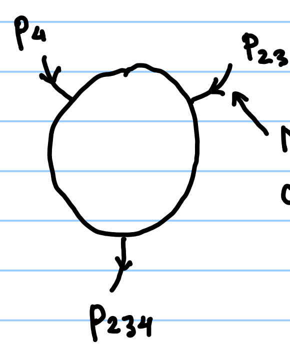
hitting p_J, p_K, $p_{J\sqcup K}$.

E.g.



(etc.)

So in our pearly tree:



We need a lift of the boundary to S^2: see in purple above.

So, $\mu^2(p_J, p_K) = \pm\, p_{J \sqcup K}$.

Claim: $\mu^2(p_i, p_j) = -\mu^2(p_j, p_i)$.

⚠ My paper is not fully justified here! Erratum at some stage...

<u>Cor</u>: $(CF^*(L,L), \mu^2) \cong \mathbb{C}[\theta_1, \dots, \theta_n]$.

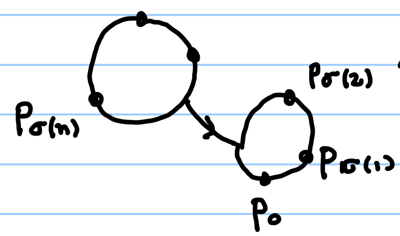
This proves ①.

Next, we want to know about higher products.

Gradings $\rightsquigarrow$ $\mu^n(\theta_{\sigma(1)}, \dots, \theta_{\sigma(n)}) = c \cdot 1$.

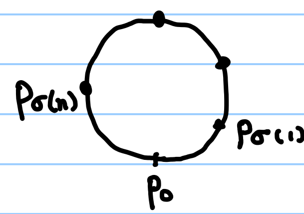
(Note: sum of H_1-gradings is $e_1 + \dots + e_n = 0$ in $\mathbb{Z}^n/\mathbb{Z}$).

What is c? It counts pearly trees:



Morse flowlines all constant, i.e. point constraints.

Topology ... $\Rightarrow$ must have



and int # $= 0$ $\Rightarrow$ deg $= n-2$.

Veronese's thm: $\exists!$ deg-$(n-2)$ curve through $n+1$ gen. points in $\mathbb{CP}^{n-2}$ (e.g. $n = 3, 4$). In fact you can solve for it explicitly.

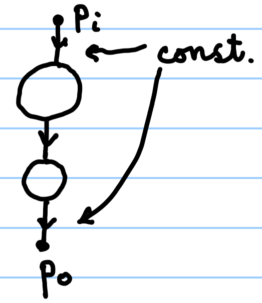
<u>Miracle</u>: it comes with a unique lift of the boundary, swapping sheets at each p_i (but not p_0); and this lift crosses each $D_i^{\mathbb{R}}$ n times, <u>always positively</u>: so the intersection number is 0!

Thus, $\mu^n(p_{\sigma(1)}, \dots, p_{\sigma(n)}) = \begin{cases} \pm 1 & \text{for one } \sigma \\ 0 & \text{for all others} \end{cases}$

This proves ②.

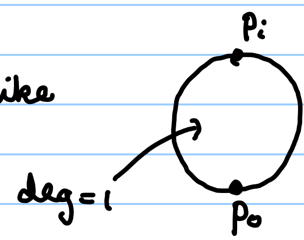
Finally, ③ : $\mu^1(p_i) = \pm r_i \cdot 1 + O(r^2)$.

Once again, this counts pearly trees:



int. # with D_i is 1, with all other D_j is 0.

$\rightsquigarrow$ has to look like



$\Rightarrow$ half a real line $\mathbb{CP}^1 \subset \mathbb{CP}^n$, through p_i and p_0. This specifies line uniquely. Then choose a half, and a lift of the boundary to S^n, swapping sheets at p_i but not at p_0 ... there's a unique choice, and it crosses all D_j positively but D_i negatively!

E.g.