Talk 1

I. Statement of the theorem

Symplectic side (A-model)

Let $X^n := \{ \sum_{i=1}^n z_i^n = 0 \} \subset \mathbb{CP}^{n-1}$

$\omega := \omega_{F.S.}|_{X^n}$

$D \subset X^n$, $D := \{ z_1 \dots z_n = 0 \}$
$= \bigcup_{i=1}^n D_i$

Let $\mathbb{K}_A := \mathbb{C}((r))$ $\quad (:= \mathbb{C}[[r]][r^{-1}])$.

To this data we will associate a $\mathbb{K}_A$ - linear A_∞ category $\mathcal{F}(X, D)$ ("the Fukaya category of X relative to the divisor D").

Our algebraic varieties:

Let $\tilde{Y}^n := \{ u_1 \dots u_n - q \cdot \sum_{i=1}^n u_i^n = 0 \} \subset \mathbb{P}^n_{\mathbb{K}_B}$

where $\mathbb{K}_B := \mathbb{C}((q))$. This is a smooth variety over $\operatorname{Spec} \mathbb{K}_B$.
You can think about it as a smooth family of complex varieties parametrised by
$q \in \operatorname{Spec} \mathbb{K}_B$:= "formal punctured disc".

Let $\tilde{G}_n := (\mathbb{Z}/n)^n / (\mathbb{Z}/n)$

embedded diagonally.

$\tilde{G}_n \circlearrowright \mathbb{P}^n_{\mathbb{K}_B}$ by multiplying u_i by nth roots of unity: but this doesn't preserve $\tilde{Y}^n$.

Observe: $\tilde{G}_n \xrightarrow{\Sigma} \mathbb{Z}/n$ well-defined

(it sends $(1,1,\dots,1) \mapsto 0$)

so we define $G_n := \ker(\Sigma) \subset \tilde{G}_n$. This acts on $\tilde{Y}^n$, and we define

$$Y^n := \tilde{Y}^n / G_n.$$

To this data we will associate a $\mathbb{K}_B$-linear A_∞ (in fact, DG) category $D^b Coh(Y^n)$.

<u>Theorem</u>: There exist

1. An isomorphism (the 'mirror map')

$$\psi : \mathbb{K}_A \xrightarrow{\sim} \mathbb{K}_B$$

$$(\text{i.e., } \psi(r) = q + a_2 q^2 + a_3 q^3 + \dots)$$

2. A quasi-equivalence of $\mathbb{K}_A$-linear A_∞ categories,

$$D^\pi \mathcal{F}(X^n, D) \cong \psi^* D^b Coh(Y^n).$$

means 'pull back the $\mathbb{K}_B$-action on hom-spaces to a $\mathbb{K}_A$-action by ψ'.

Think: LHS is a family of $\mathbb{C}$-linear A_∞ cats parametrised by $r \in$ formal punctured disc, RHS " " " $q \in$ " ", and they only match up when you say how the formal punctured discs Spec $\mathbb{K}_A$ and Spec $\mathbb{K}_B$ are identified.

II. The exact Fukaya category, $\mathcal{F}(M)$

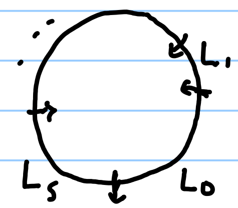
Recall $\mathcal{F}(M)$ is a $\mathbb{C}$-linear A_∞ category

- objects are exact Lag. branes $L \subset M$.

- morphisms are $CF^*(L_0, L_1) := \mathbb{C}\langle L_0 \cap L_1 \rangle$

- structure maps μ^s count pseudoholomorphic discs

$$u : (\mathbb{D}, \partial \mathbb{D}) \longrightarrow M$$



- it is $\mathbb{Z}$-graded.

If we equip our Lagrangians L with a lift $\tilde{L}$ to $\tilde{M} :=$ universal abelian cover of M, then we can equip hom-spaces with an extra $H_1(X \setminus D)$ grading:

$x \in L_0 \cap L_1$

$\Rightarrow$ $\deg(x) :=$ the element of H_1 s.t. x lifts to $\tilde{x} \in \tilde{L}_0 \cap \deg(x) \cdot \tilde{L}_1$.
cov. gp. action

Then because our hol. discs are contractible, they lift to $\tilde{M}$

$$\Rightarrow \deg(\mu^s(x_1, \dots, x_s)) = \sum_i \deg(x_i).$$

Here is a better formulation:

$$\begin{tikzcd} & \mathcal{G}M \arrow[d] \\ L \arrow[r] \arrow[ur, "TL"] & M \end{tikzcd}$$

$:=$ Lag. Grassmannian bundle

An <u>anchored</u> Lag. brane is one equipped with a lift to $\tilde{\mathcal{G}} M$:= univ. ab. cov. of $\mathcal{G}M$.

(this replaces 'graded' Lagrangians, which come with a lift to a choice of fibrewise univ. cover of $\mathcal{G}M$).

This equips $CF^*(L_0, L_1)$ with a $H_1(\mathcal{G}M)$-grading. We have

$$\begin{tikzcd} \mathcal{G}_p M \arrow[r, hook] & \mathcal{G}M \arrow[d] \\ & M \end{tikzcd}$$

$$\Rightarrow \underset{\substack{\cong \\ \mathbb{Z}}}{H_1(\mathcal{G}_p M)} \xrightarrow[i_*]{} H_1(\mathcal{G}M) \longrightarrow H_1(M)$$

The A_∞ structure maps respect this grading: μ^s has degree $i_*(2-s)$ (N.B. this maps to $0 \in H_1(M)$, c.f. above).

<u>Defn</u>: A <u>grading datum</u> $\mathcal{G}$ is an abelian group Y, with a map $i: \mathbb{Z} \to Y$, and a map $\sigma: Y \to \mathbb{Z}/2$, s.t. $\sigma \circ i(1) = 1$.

A $\mathbb{G}$-graded A_∞ alg. is Y-graded, so the A_∞ maps μ^s have degree $2-s$, and satisfy the A_∞ rel'ns with signs det'd via σ. A $\mathbb{G}$-graded A_∞ cat. is the same, where there is an action of Y on objects by 'shift', with the expected properties.

E.g. $\mathcal{F}(M)$ is $\mathbb{G}(M)$-graded, where

$$\mathbb{G}(M) := \Big(\underset{\mathbb{Z}}{\overset{\cong}{H_1(\mathcal{G}_p M)}} \xrightarrow{i} H_1(\mathcal{G}M) \xrightarrow{\sigma} \mathbb{Z}/2\Big)$$

w_1 (universal bundle)

III. Covers

A map of grading data $p: \mathbb{G}_1 \to \mathbb{G}_2$:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i_1"] \arrow[d, equal] & Y_1 \arrow[r, "\sigma_1"] \arrow[d, "p"] & \mathbb{Z}/2 \arrow[d, equal] \\ \mathbb{Z} \arrow[r, "i_2"'] & Y_2 \arrow[r, "\sigma_2"'] & \mathbb{Z}/2 \end{tikzcd}$$

E.g. If $\pi: N \to M$ is a cover,

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, "="] & H_1(\mathcal{G}N) \arrow[r] \arrow[d, "\pi_*"] & \mathbb{Z}/2 \arrow[d, "="] \\ \mathbb{Z} \arrow[r] & H_1(\mathcal{G}M) \arrow[r] & \mathbb{Z}/2 \end{tikzcd}$$

Defn: If $\mathcal{C}$ is a $\mathbb{G}_2$-graded A_∞ cat. and

$p: \mathbb{G}_1 \to \mathbb{G}_2$

is a map of grading data, set

$p^* \mathcal{C}$ = a G_1-graded A_∞ cat. with

- same objects
- $\hom_{p^*\mathcal{C}}(A,B)_y := \hom_{\mathcal{C}}(A,B)_{p(y)}$
- the induced A_∞ structure maps.

(Y_1 acts on objects via $p : Y_1 \to Y_2$).

Lem: If $\pi : N \to M$ is a cover, there's a fully faithful embedding

$$\pi^* \mathcal{F}(M) \hookrightarrow \mathcal{F}(N).$$

Pf: We have

$$\begin{tikzcd}
 & N \arrow[d] & \tilde{N} \arrow[l] \arrow[d, equal] \\
L \arrow[r] \arrow[rr, bend right, "\tilde{L}"'] & M & \tilde{M} \arrow[l]
\end{tikzcd}$$

$$\Rightarrow \quad \mathrm{Ob}\, \mathcal{F}(M) \not\subset \mathrm{ob}(\mathcal{F}(N))$$

$\subseteq$

if L is embedded in N

but image is immersed in M.

$$\hom_{\pi^*\mathcal{F}(M)}(L_0, L_1)_y := \hom_{\mathcal{F}(M)}(L_0, L_1)_{\pi_* y}$$

$$= \langle \text{those intersection points that lift to } N \rangle$$

$$= \hom_{\mathcal{F}(N)}(\tilde{L}_0, \tilde{L}_1).$$

The J-holomorphic discs in M lift to N, as they're contractible $\Rightarrow$ A_∞ products in $\pi^* \mathcal{F}(M)$ coincide with those in $\mathcal{F}(N)$.

IV. The relative Fukaya category

$D \subset X$ smooth normal-crossings, $D = \bigcup_{i=1}^{k} D_i$

Assume for simplicity: each D_i is P.D. to $[\omega]$ $\Rightarrow$ $X \setminus D$ is exact, can define $\mathcal{F}(X \setminus D)$, it's $\mathbb{G}(X \setminus D)$-graded.

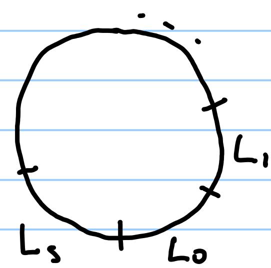
Defn: The relative Fukaya category $\mathcal{F}(X, D)$ has

- the same objects as $\mathcal{F}(X \setminus D)$
- morphism spaces

$$\hom(L_0, L_1) := R\langle L_0 \cap L_1 \rangle$$

where $R := \mathbb{C}[[r_1, \dots, r_k]]$

- A_∞ structure maps μ^s count holomorphic discs



$u: \mathbb{D} \to X$

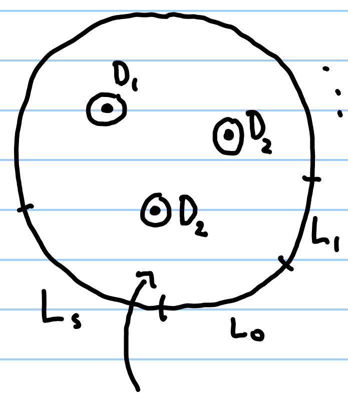
weighted by

$$r^{u \cdot D} := r_1^{u \cdot D_1} \cdots r_k^{u \cdot D_k} \in R$$

($u \cdot D_i \geq 0$ because holom. curves hit complex divisors positively).

discs no longer contained in $X \setminus D$
$\Rightarrow$ boundary no longer vanishes in $H_1(X \setminus D)$
$\Rightarrow$ grading gets messed up.

But, if we equip r_i with degree = class of a meridian curve around D_i, then



defines a homology between
$\deg(\mu^s(x_1, \dots, x_s)) - \deg(x_1) - \dots - \deg(x_s)$

and $\deg(r_1 r_2^2)$

$\Rightarrow$ μ^s still preserves $H_1(X \setminus D)$-degree.

Similarly, if we equip r_i with degree $\in H_1(\mathcal{G}(X \setminus D))$ given by a lift of the meridian loop to $\mathcal{G}(X \setminus D)$, so that the lift extends across a small disc hitting D_i once, then $\mathcal{F}(X, D)$ is $\mathcal{G}(X \setminus D)$-graded.

To get a category over $\mathbb{C}((r))$ in the theorem, we tensor $\mathcal{F}(X,D) \otimes_R \mathbb{C}((r))$, where $R \to \mathbb{C}((r))$, $r_i \mapsto r$.

V. Branched covers

Suppose $\pi : (X', D') \to (X, D)$

is a branched cover, with branching of degree a_i about D_i. So

$$\pi|_{X' \setminus D'} : X' \setminus D' \longrightarrow X \setminus D$$

is an unbranched cover, and induces

$$\pi_* : \mathcal{G}(X' \setminus D') \to \mathcal{G}(X \setminus D)$$

as before. We have

$$\pi^* \mathcal{F}(X \setminus D) \hookrightarrow \mathcal{F}(X' \setminus D')$$

like before, but

$$\pi^* \mathcal{F}(X, D) \not\hookrightarrow \mathcal{F}(X', D'):$$

holomorphic discs $u : \mathbb{D} \to X$ that intersect D will not lift to X'!

But it's clear what to do: we must count holom. discs $u : \mathbb{D} \to X$ that are tangent to D_i to order $a_i - 1$ whenever they meet.

I.e., we treat X as an orbifold

$$X' / \text{cov. gp.},$$

and count holom. maps into this orbifold.

This defines the orbifold relative Fukaya category $\mathcal{F}(X, D^{\underline{a}})$.

There is a fully faithful embedding

$$\pi^* \mathcal{F}(X, D^{\underline{a}}) \hookrightarrow \mathcal{F}(X', D').$$

E.g.

$$\begin{array}{ccl}
X^n & = & \{ \sum_i z_i^n = 0 \} \subset \mathbb{CP}^{n-1} \\
\cup & & \\
D = \bigcup_i \{z_i = 0\} & \Bigg\downarrow & \begin{array}{c} [z_1 : \ldots : z_n] \\ \downarrow \\ [z_1^n : \ldots : z_n^n] \end{array} \\
\cap & & \\
\mathbb{CP}^{n-2} & = & \{ \sum_i z_i = 0 \} \subset \mathbb{CP}^{n-1}
\end{array}$$

In order to understand $\mathcal{F}(X^n, D)$, it suffices to understand $\mathcal{F}(\mathbb{CP}^{n-2}, D^{\underline{n}})$.

This will be our next topic.