Note Title 12/06/2015

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Matrix factorizations / B-model

Recall:

$A := \mathbb{C}[\partial_1, \dots, \partial_n]$

$R := \mathbb{C}[[Q_1, \dots, Q_n]]$

Both graded in $\mathbb{Z} \oplus \mathbb{Z}^n / 2(n-1) \oplus (1, \dots, 1) = G$

$\mathcal{A} := A \otimes R$

A_∞ structure on $\mathcal{A}$ uniquely characterized by

1. Grading
2. $\mathrm{Sym}(n)$-equivariant
3. μ_0^2 = exterior product
4. $\Phi_{HKR}(\mu^{\geq 3}) = \pm z_1 \cdots z_n \pm \sum Q_i z_i^n + O(Q^2)$. $\in (HH^\cdot(A) \otimes R)^2$

We consider

$S := R[z_1, \dots, z_n]$

$W \in S$, $W := z_1 \cdots z_n + \sum Q_i z_i^n$

S has a $\mathbb{G}$-grading; W has deg 2.

$\deg(Q_i) = 2(1-n) \oplus (0,\dots,\underset{i}{n},0,\dots,0)$

$\deg(z_i) = 2 \oplus (0,\dots,-1,0,\dots,0)$

$\deg(W) = 2n \oplus (-1,\dots,-1)$

$= 2n + 2(1-n) \oplus 0$

$= 2 \oplus 0$.

Now $MF^{\mathbb{G}}(S,W)$:

Obj := Free $\mathbb{G}$-graded S-modules X

with $d_X : X \circlearrowleft$ of deg $= 1$,

$d_X^2 = W \cdot \mathrm{id}$

Lena's talk: construct a MF (B, d_B)

$B := S[\theta_1,\dots,\theta_n]$

$d_B := \sum z_i \frac{\partial}{\partial\theta_i} + w_i \theta_i$

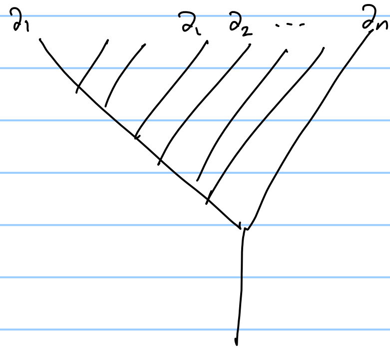
(where $\sum_i z_i w_i = W$)

$\rightsquigarrow \mathrm{End}(B, d_B) \xrightarrow[A_\infty]{\sim} \big(R[\partial_1,\dots,\partial_n], \underset{\substack{\| \\ \text{exterior} \\ \text{product}}}{\mu^2}, \mu^3, \dots\big)$

<u>Lem:</u> $\Phi_{HKR}(\mu^{\geq 3}) = W \in R[z_1,\dots,z_n][\partial_1,\dots,\partial_n]$

Sketch:

$$\mu^{i_1+\dots+i_n}(\partial_1,\dots,\partial_1,\dots,\partial_n,\dots,\partial_n) = (\partial^{\underline{i}} W)(0)$$



contributions from other trees vanish.

$$W = z_1 \cdots z_n + \sum Q_i z_i^n$$

$$\mu^n(\partial_1,\dots,\partial_n) = 1$$

<u>Cor.</u> $\mathcal{B} := \mathrm{End}_{MF^G}(B, d_B)$ satisfies the conditions required by our "recognition principle" from yesterday:

- $\mathbb{G}$-graded
- $\mathrm{Sym}(n)$-eq
- μ_0^2 = ext prod
- $\Phi_{HKR}(\mu^{\geq 3}) = W + O(Q^2)$

Cor. $\exists\ \psi : R \xrightarrow{\sim} R$ and

A_∞ quasi-isomorphism

$$\mathcal{A} \cong \psi^* \mathcal{B} \cong \mathrm{End}_{MF^G(S,W)}(\mathcal{B}, d_{\mathcal{B}})$$

$$\mathcal{A} = \mathrm{End}_{\mathcal{F}(\mathbb{CP}^{n-2},\, D^n)}$$

$$\Longrightarrow \tilde{\mathcal{A}} = \psi^* \tilde{\mathcal{B}} \leftarrow p^*\mathcal{B} \subset p^* MF^G(S,W) \overset{?}{\wr} D^bCoh^G(\tilde{Y}^n)$$

$$q_* p^* \mathcal{A} \subset \mathcal{F}(X^n, D)$$

all shifts

$$p : H_1(\mathcal{G}(X^n \setminus D)) \longrightarrow H_1(\mathcal{G}\, P^{n-2})$$

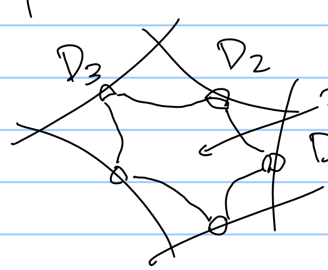
$$H_1(\mathcal{G}(X^n \setminus D)) \cong \mathbb{Z} \oplus \mathbb{Z}^n/(1,\dots,1) \xrightarrow{q} \mathbb{Z}$$

$$\pi : X^n \setminus D \longrightarrow \mathbb{CP}^{n-2} \setminus D$$

$$\pi_* : H_1(\mathcal{G}(X^n \setminus D)) \xrightarrow{p} H_1(\mathcal{G}(\mathbb{CP}^{n-2} \setminus D))$$

$$\pi_* = p$$

$$H_1(\mathcal{G}(\mathbb{CP}^{n-2} \setminus D)) \leftarrow \mathbb{Z} \oplus \mathbb{Z}^n / 2(1-n) \oplus (1,\dots,1)$$



2-chain gives relation

loop in Gr $\to$ Log

each summand from section restricted to bdry of disk

D_i (#)

$$p : \mathbb{Z} \oplus \mathbb{Z}^n/(1,\dots,1) \longrightarrow \mathbb{Z} \oplus \mathbb{Z}^n / 2(1-n) \oplus (1,\dots,1)$$

$$j \oplus (j_1,\dots,j_n) \longmapsto j + 2(1-n)(j_1+\dots+j_n) \oplus (nj_1,\dots,nj_n)$$

$$\mathrm{coker}(p) \cong (\mathbb{Z}/n)^n/(1,\dots,1) = \text{covering group}$$

$$\ker(p) = 0$$

Defn: $G_{MF(n)} := \mathbb{Z} \oplus \mathbb{Z} / _{2 \oplus -n}$

Lemma There's a commutative diagram of grading data:

$$\begin{tikzcd}
\mathbb{Z} \oplus \mathbb{Z}^n / _{(1,\dots,1)} \arrow[r, "q"] \arrow[d] & \mathbb{Z} \arrow[d, "f"] \\
\mathbb{Z} \oplus \mathbb{Z}^n / _{2(1-n) \oplus (1,\dots,1)} \arrow[r, "g"'] & G_{MF(n)}
\end{tikzcd}$$

Lem: If V is $\left(\mathbb{Z} \oplus \mathbb{Z}^n / _{2(1-n) \oplus (1,\dots,1)} \right)$-graded, then $f^* g_* V$ admits a Γ-grading,

$$\Gamma := (\mathbb{Z}/n)^n / _{(1,\dots,1)}$$

$\longleftrightarrow$ Γ^*-action, $\Gamma^* := \mathrm{Hom}(\Gamma, \mathbb{C}^*)$

And $(f^* g_* V)^{\Gamma^*} \cong q_* p^* V$

Cor. $q_* p^* MF_G(S,W) \cong \left(f^* g_* MF_G(S,W) \right)^{\Gamma^*}$

Defn: Let S be a $\mathbb{Z}$-graded ring, $W \in S$ homogeneous of $\deg = n$

(e.g. $S = \mathbb{k}[z_1, \dots, z_n]$, $\mathbb{Z}$-grading = degree)

Equip S with a $\Gamma_{MF(n)}$-grading, by putting S_j in degree $0 \oplus j \in \mathbb{Z} \oplus \mathbb{Z}/_{2\oplus -n}$.

We define

$$\mathrm{Gr}\,MF(S,W) := f^* MF^{\Gamma_{MF(n)}}(S,W)$$

<u>Thm.</u> (Orlov):

If $S = \mathbb{K}[z_1, \dots, z_n]$

$$\left.\begin{array}{l} W \in S \ \deg = n \\ \{W=0\} \subset \mathbb{P}^{n-1}_{\mathbb{K}} \text{ smooth} \end{array}\right\} \Longrightarrow H^0(\mathrm{Gr}\,MF(S,W)) \cong D^b\mathrm{Coh}(\{W=0\})$$

$$R = \mathbb{C}[[Q_1, \dots, Q_n]] \longrightarrow \mathbb{C}((Q))$$
$$Q_i \longmapsto Q$$

$$\begin{array}{ccc} \mathcal{F}(X^\vee, D) \otimes_R \mathbb{C}((Q)) & & \mathrm{Gr}\,MF^{\Gamma^*}(S \otimes_R \mathbb{C}((Q)), W \otimes 1) \cong D^b\mathrm{Coh}^{\Gamma^*}(\{W=0\}) \\ \cup & & \cup \\ \tilde{A} \otimes_R \mathbb{C}((Q)) & \xrightarrow{\sim} & \psi^*(\tilde{\mathcal{B}} \otimes_R \mathbb{C}((Q))) \end{array}$$