Note Title 11/06/2015

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Deformation theory

Grading: $\mathbb{Z} \oplus \mathbb{Z}^n / 2(1-n) \oplus (1,\dots,1)$

Ring $R = \mathbb{C}[[Q_1,\dots,Q_n]]$

Vec space (alg) $A = \mathbb{C}[\theta_1,\dots,\theta_n]$

$$|\theta_i| = -1 \oplus (0 \dots \underset{i}{1} \dots 0)$$

$R_0 \cong \mathbb{C}[[T]] \quad T := Q_1 \cdots Q_n$

$$\Phi_{HKR} : CC^*(A) \longrightarrow \mathbb{C}[[z_1,\dots,z_n]][\theta_1,\dots,\theta_n]$$
$$\alpha \longmapsto \sum_d \alpha^d(\underline{z},\dots,\underline{z})$$

$(|z_i| = 2 \oplus (0,\dots,-1,\dots,0))$ where $\underline{z} := \sum z_i \theta_i$

Thm. If $\mu^s : (A \otimes R)^{\otimes_R s} \longrightarrow A \otimes R$ is an R-linear A_∞ struc., and

1. it is G graded
2. it is $\mathrm{Sym}(n)$-equivariant
3. If $\mu^\bullet = \mu^\bullet_0 + \mu^\bullet_1 + \dots$ expansion in powers of Q

then $\mu^\bullet_0 = $ ext product

④ $\Phi_{HKR}(\mu^{\geq 3}) = \pm z_1 \cdots z_n \pm \sum Q_i z_i^n + O(Q^2)$

If $\mu^\cdot$ and $m^\cdot$ are two such, then there exist

ⓐ $\varphi \in \mathbb{C}[[T]]$, $\varphi(0) = 1$

ⓑ An R-linear A_∞ quasi-isom

$$F : (A \otimes R, \mu^\cdot) \longrightarrow \psi^*(A \otimes R, m^\cdot)$$

where $\psi : R \xrightarrow{\sim} R$

$$Q_i \longmapsto Q_i \cdot \varphi(T)$$

Pf: Work order-by-order in Q_i.

First, order 0:

$$F_0 : (A, \mu_0^\cdot) \longrightarrow (A, m_0^\cdot)$$

Grading $\Rightarrow$ $\mu_0^s = m_0^s = 0$ unless $s \equiv 2 \pmod{n-2}$

$\Longrightarrow \mu_0^s = 0$ for $2 < s < n$

$\Longrightarrow [\mu_0^n] = \alpha b^n \in HH^2(A)^{2-n}$

Lem $HH^2(A)^{2-s} \cong \begin{cases} \mathbb{C} \cdot z_1 \cdots z_n & \text{if } s=n \\ 0 & \text{else} \end{cases}$

Note: $ob^n_{\mu_0} = ob^n_{m_0} = z_1 \cdots z_n$

$\Rightarrow$ By Nati's talk, $\exists$ an A_∞ quasi-iso

$$F_0 : (A, \mu_0^\bullet) \longrightarrow (A, m_0^\bullet)$$

Note: $[\mu_0^\bullet + \mu_1^\bullet + \cdots, \mu_0^\bullet + \mu_1^\bullet + \cdots] = 0$

1st order expansion in Q

$\Rightarrow [\mu_0^\bullet, \mu_1^\bullet] = 0$

$\Rightarrow [\mu_1^\bullet] \in HH^\bullet(A, \mu_0^\bullet)$

1st order def class

If $m := \langle Q_1, \ldots, Q_n \rangle \subset R$

$$[\mu_1^\bullet] \in \left(\left(HH^\bullet(A, \mu_0^\bullet) \otimes m/m^2\right)^2\right)^{Sym(n)}$$

In fact, $[\mu_1^\bullet]$ generates as $\mathbb{C}$ vec sp.

In fact, $[\mu_1]^\bullet$ generates

$$\left(HH^\bullet(A, \mu_0^\bullet) \otimes R\right)^{2, Sym(n)}$$

as $\mathbb{C}[[T]]$-module.

Pf: $[\mu_1^\circ] = \pm \sum_i Q_i z_i^n$

(to leading order in length filtration)

$F_{\geq s} CC^\circ(A) := \prod_{d \geq s} \mathrm{Hom}(A^{\otimes d}, A)$

$\Rightarrow$ spec. seq. with E_1 page $HH^\circ(A, \mu_0^2)$

converges to $HH^\circ(A, \mu_0^\circ)$

killed on E_n page

because $= d\delta^n_{\mu_0^\circ}$

$\left(\mathbb{C}[z_1, \dots, z_n][\theta_1, \dots, \theta_n] \otimes R\right)^{2, Sym(n)} = \langle \cancel{z_1, \dots z_n}, \sum Q_i z_i^n \rangle$

Note: $2 := (2,0) \in \mathbb{Z} \oplus \mathbb{Z}^n / 2(1-n) \oplus (1, \dots, 1)$

differential $= 0$

Prop. If μ°, m° are two minimal

A_∞ str. s.t. $\mu_0 = m_0$

and s.t. $[\mu_1] = [m_1]$

generates $HH^2(A) \otimes \mathbb{C}[[T]]$

over $\mathbb{C}[[T]]$.

Then we can construct

1) $\psi : \mathbb{C}[[T]] \to \mathbb{C}[[T]]$

2) $F : (A \otimes \mathbb{C}[[T]], \mu) \xrightarrow{\sim} (A \otimes \mathbb{C}[[T]], m^\circ)$

Pf: Constructing F order-by-order in T,

$F_0 = Id$. Inductively, suppose

$$\sum F(\dots \mu^i(\ )\dots) = \sum m^i(F^\cdot(\dots),\dots,F^\cdot(\dots))$$

up to order d.

Then to make it true to order $d+1$

$$\Rightarrow \sum F_{\le d}(\dots \mu(\dots)\dots) - \sum m(F_{\le d}\dots F_{\le d}\dots)$$

$$+ \underbrace{F_{d+1}(\dots \mu_0(\dots)\dots) + \underset{\mu_0}{\overset{\shortparallel}{m_0}}(\dots F_{d+1}(\dots)\dots)}_{\partial F_{d+1}} = 0$$

$$= T^{d+1}\cdot [\alpha_{d+1}] = T^{d+1}\cdot c_{d+1}[\mu_1]$$

Make substitution: $T \mapsto T + c_{d+1} T^{d+1}$

Prop. $\mathcal{A} := CF^*_{\mathcal{F}(\mathbb{CP}^{n-2}, D^n)}(L,L)$

has the properties in the thm.

Pf: (1) G-graded ✓

(2) $Sym(n)$ - equivariant

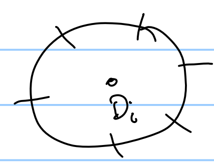
(3) $\mu_0^2 =$ ext prod ✓

(4) $\mu^n(z,\dots,z) = \pm z_1 \dots z_n$

$$\mu^1(\Theta_i) = \pm Q_i \cdot 1 \quad \text{in } \mathcal{F}(\mathbb{CP}^{n-2}, D)$$

$$\Phi_{HKR}(\mu^1_1) = \sum_i Q_i z_i$$

<u>Prop.</u> Suppose $\alpha_i := [\mu_{1,i}] \in HH^\cdot(\mathcal{F}(X\setminus D))$ is the i^{th} 1^{st}-order def. class given by $\mathcal{F}(X, D)$, i.e. counting disks that <u>only</u> hit D_i.



Let $\beta_i := [\mu_{1,i}] \in HH^\circ(\mathcal{F}(X\setminus D))$ come from ...

$\mathcal{F}(X, D^a)$

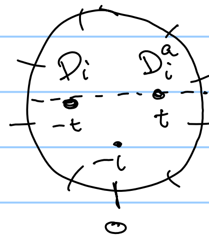
Then $\beta_i = \alpha_i^a$ ← w.r.t. Yoneda product on HH^*

<u>Proof</u>: First:

$$F \cup G = \sum \mu_0(\dots F^\circ(\dots)\dots G^\circ(\dots)\dots)$$

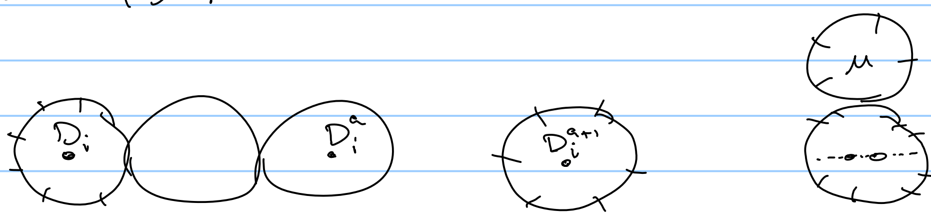
Yoneda product

Now, consider moduli space $\mathcal{M}$:



defines an element $H_a \in CC^*(\mathrm{Fuk})$

Gromov compact-n of 1-dim'l comp't ∂ is:



$t \to 1$ $\quad$ $t \to 0$ $\quad$ $0 < t < 1$

$$\alpha_i \cup \beta_{i,a} + \beta_{i,a+1} + [\mu_0, H_a] = 0$$

$$\Rightarrow \beta_{i,a} = \alpha_i^a \quad \text{by induction}$$

$\square$