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I. Statement

Symplectic (A-model)

$$X^n := \left\{ \sum_{i=1}^{n} z_i^n = 0 \right\} \subset \mathbb{CP}^{n-1}$$

$$\omega := \omega_{FS}\Big|_{X^n}$$

$D \subset X^n$, $\quad D = \{ z_1 \cdots z_n = 0 \} = \bigcup_{i=1}^{n} D_i$

Let $\mathbb{K}_A = \mathbb{C}((Q))$

To (X^n, D) we associate $\mathcal{F}uk(X^n, D)$

- a $\mathbb{K}$-linear A_∞ cat.

Algebraic (B-model)

$$\tilde{Y}^n := \left\{ u_1 \cdots u_n - q \cdot \sum_{i=1}^{n} u_i^n = 0 \right\} \subset \mathbb{P}^{n-1}_{\mathbb{K}_B}$$

$\mathbb{K}_B = \mathbb{C}((q))$

$$\tilde{G}^n := (\mathbb{Z}/n)^n / (\mathbb{Z}/n) \circlearrowright \mathbb{P}^{n-1}_{\mathbb{K}_B}$$

$$\tilde{G}^n \xrightarrow{\Sigma} \mathbb{Z}/n \qquad \text{(sum coordinates)}$$

$$G^n := \ker(\Sigma) \circlearrowright \tilde{Y}^n$$

$Y^n := \tilde{Y}^n / G_n$

To this we associate a

$\mathbb{K}_B$-linear A_∞ (actually D.G.: $\mu^{\geq 3} = 0$) cat.,

$D^b Coh(Y^n) := D^b Coh^{G_n}(\tilde{Y}_n)$

<u>Thm:</u> There exist

① An iso $\psi : \mathbb{K}_A \xrightarrow{\sim} \mathbb{K}_B$

$$\psi(Q) = q + c_2 q^2 + \cdots$$

② An A_∞ quasi-equivalence of $\mathbb{K}_A$-linear A_∞ cats

$$D^\pi Fuk(X, D) \simeq \psi^* D^b Coh(Y^n)$$

$$\left(\underset{?}{\Longrightarrow} D^\pi Fuk(X) \cong \hat{\psi}^* D^b Coh(Y^n \otimes_{\mathbb{K}_B} \Lambda) \right)$$

II. <u>The relative Fukaya category</u>

(X, ω) - compact sympl manifold

$D \subset X$ divisor $PD([D]) = [\omega]$

$\Rightarrow X \setminus D$ exact

$\Rightarrow$ Can define $\mathcal{F}(X \setminus D)$ over $\mathbb{C}$.

Obj = exact $L \subset X \setminus D$ (compact)

$$\mathrm{Mor} = \mathbb{C}\langle L_0 \cap L_1 \rangle$$

$$\mathrm{Maps} = \text{count } u : (\mathbb{D}, \partial) \underset{\text{holom}}{\longrightarrow} (X \setminus D, L_i)$$

The relative Fukaya category:

$\mathcal{F}(X, D)$ defined over $\mathbb{C}[[Q_1, \dots, Q_n]] =: R$

one variable for each component of $D = \bigcup_{i=1}^{n} D_i$

Obj = same as $\mathcal{F}(X \setminus D)$

$\mathrm{Mor} = R\langle L_0 \cap L_1 \rangle$

Maps = count $u : (\mathbb{D}, \partial) \longrightarrow (X, L_i)$,

weighted by $Q_1^{u \cdot D_1} \cdots Q_n^{u \cdot D_n} \in R$

$\mathcal{F}(X, D)$ is a deformation of $\mathcal{F}(X \setminus D)$ over R.

$$\mathcal{F}(X, D) \otimes_R \mathbb{C} \simeq \mathcal{F}(X \setminus D).$$

We will try to understand how $\mathcal{F}(X, D)$ behaves w.r.t. branched covers:

$$\pi : (Y, E) \longrightarrow (X, D) \rightsquigarrow \mathcal{F}(Y, E) \overset{?}{\leftrightarrows} \mathcal{F}(X, D)$$

branched along E

E.g.

$$\begin{tikzcd} X^n = \{\sum z_i^n = 0\} \arrow[d] & E = \{z_1 \cdots z_n = 0\} \\ \mathbb{CP}^{n-2} = \{\sum y_i = 0\} & D = \{y_1 \cdots y_n = 0\} \end{tikzcd}$$

$$[z_1 : \cdots : z_n] \mapsto [z_1^n : \cdots : z_n^n]$$

First, about gradings.

exact manifold, e.g. $X \setminus D$

We equip $L \subset M$ with a grading:

$$\begin{tikzcd} & \widetilde{GM} \arrow[d] \\ & GM \arrow[d] \\ L \arrow[uur, dashed, "\text{grading}"] \arrow[ur] \arrow[r, hook] & M \end{tikzcd}$$

$\widetilde{GM}$ - universal abelian cover

GM = Lagr. Grassmannian

Lagrangian Grassmannian of T_mM

We have

$$\begin{tikzcd} G_mM \arrow[r, hook] & GM \arrow[d] \\ m \arrow[r] & M \end{tikzcd}$$

$$\Rightarrow H_1(G_mM) \longrightarrow H_1(GM) \longrightarrow H_1(M)$$

We can equip each $x \in L_0 \cap L_1$ with a degree in $H_1(GM)$.

And $\mathcal{F}(M)$ becomes $H_1(GM)$ graded

I.e. μ^d has degree $2-d \in \mathrm{Im}(\mathbb{Z}) \subset H_1(GM)$

$$CF^*(L_0,L_1) \cong \bigoplus_{\gamma \in H_1(\mathcal{G}M)} CF^\gamma(L_0,L_1)$$

An unbranched abelian cover of exact sympl. mflds $\pi: M \to N$ induces

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, equal] & H_1(\mathcal{G}M) \arrow[d, "p:=\pi_*"] \\ \mathbb{Z} \arrow[r] & H_1(\mathcal{G}N) \end{tikzcd}$$

We have $Ob(Fuk(M)) \cong Ob(Fuk(N))$

$$\begin{tikzcd} & \tilde{\mathcal{G}N} \arrow[r, equal] \arrow[d] & \tilde{\mathcal{G}M} \arrow[d] \arrow[r] & \check{M} \arrow[dl] \\ L \arrow[r, hook] \arrow[ur] & N & M \arrow[l] & \end{tikzcd}$$

However, $CF^*_{Fuk(M)}(L_0,L_1) \not\cong CF^*_{Fuk(N)}(L_0,L_1)$.

Rather, $CF^*_{Fuk(M)}(L_0,L_1) \cong \bigoplus_{\gamma \in H_1(\mathcal{G}M)} CF^\gamma_{Fuk(M)}(L_0,L_1)$

$$\cong \bigoplus_{\gamma \in H_1(\mathcal{G}M)} CF^{\pi_*(\gamma)}_{Fuk(N)}(L_0,L_1)$$

<u>Def-n</u> A grading datum $\mathbb{G}$ is:

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[dr] & Y \arrow[d] \\ & \mathbb{Z}/2 \end{tikzcd}$$

Y = ab. gp

Defn: A Y-graded A_∞ cat. is one with $\hom^*(L_0,L_1) \cong \bigoplus_{y\in Y} \hom^y(L_0,L_1)$ and μ^d of degree $\mathrm{im}(2-d)$ (signs come from $\mathbb{Z}/2$)

If have

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & Y_1 \arrow[d, "p"] \\ \mathbb{Z} \arrow[r] & Y_2 \end{tikzcd}$$

and $\mathcal{C}$ a $(\mathbb{Z}\to Y_2)$-graded A_∞-cat, then we can define

$p^*\mathcal{C}$: - same objects

- $\hom^y_{p^*\mathcal{C}}(L_0,L_1) := \hom^{p(y)}_{\mathcal{C}}(L_0,L_1)$

- induced A_∞ maps

Lem: $\mathrm{Fuk}(M) \cong p^*\,\mathrm{Fuk}(N)$,

where $p: H_1(\mathcal{G}M) \longrightarrow H_1(\mathcal{G}N)$

$p := \pi_*$.