# Membrez II

Note Title 11/06/2015

## Abouzaid's split-generation criterion

$(M, \omega)$ Liouville manifold

$\mathcal{W}$ subcategory of $WF(M)$ with finite number of objects

**Thm (Abouzaid)** $\mathcal{B}$ a full subcategory of $\mathcal{W}$. If the image of

$$HH_*(\mathcal{B}, \mathcal{B}) \to HH_*(\mathcal{W}, \mathcal{W}) \xrightarrow{H^*(OC)} SH^*(M)$$

hits $Id \in SH^*(W)$, then $\mathcal{B}$ split-generates $\mathcal{W}$.

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## Wrapped Fukaya category $WF(M)$

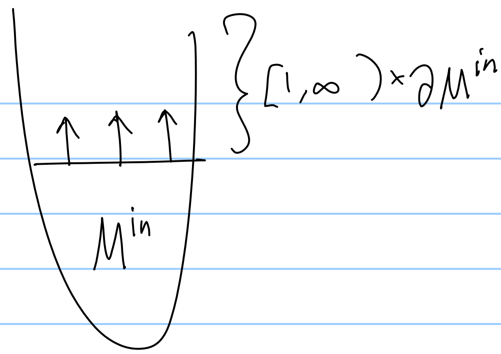
$(M, \omega = d\lambda)$ Liouville manifold with cylindrical end,

$$M = M^{in} \cup_{\partial M^{in}} (\partial M^{in} \times [1, \infty))$$

$Z_\lambda$ v field

$$i_{Z_\lambda} \omega = \lambda$$

$Z_\lambda = r\partial_r$ on cylindrical end.




$(\partial M^{in}, \overset{\alpha}{\overset{\shortparallel}{\lambda|_{\partial M^{in}}}})$
contact

$\psi^\rho$ Liouville flow (flow of $Z_\lambda$)
for time $\log \rho$.

$\psi^\rho(r,x) = (\rho r, x)$ in cylindrical end.

<u>Lagrangians</u>: $L \subset M$, exact, cylindrical at $\infty$.
($\lambda|_L = df_L$, $df_L$ has compact support)

$H$ admissable Ham, $H: M \to \mathbb{R}$, $H(r,x) = r^2$
outside a compact set

$X_H$ Ham v. field
($dH = \omega(X_H, \cdot)$)

$X_H = 2r R_\alpha$ ($R_\alpha$ Reeb v. field of contact form $\alpha$)
outside compact set

$\phi_H^1$ – time 1 flow.

$L_0, L_1$, $\mathcal{X}(L_0,L_1) = \left\{ \gamma : [0,1] \to M \;\middle|\; \begin{array}{l} \dot\gamma = X_H \circ \gamma \\ \gamma(0) \in L_0, \gamma(1) \in L_1 \end{array} \right\}$

Wrapped Floer complex:

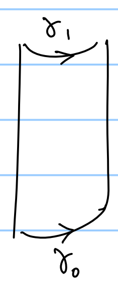
$$CW(L_0, L_1, H) = \bigoplus_{\gamma \in \mathcal{X}(L_0, L_1)} \mathbb{K}_\gamma$$

$$\mu^1 : CW(L_0, L_1, H) \circlearrowleft$$

$$J_t \in \mathcal{J}(M), \qquad t \in [0,1]$$

$$\mathcal{M}(\gamma_0, \gamma_1) := \left\{ u : \mathbb{R} \times [0,1] \to M \;\middle|\; \begin{array}{l} u(s,0) \in L_0 \\ u(s,1) \in L_1 \\ \partial_s u + J_t \partial_t u = X_H \Leftrightarrow (du - X_H \otimes dt)^{0,1} = 0 \\ u \xrightarrow[s \to -\infty]{} \gamma_0 \\ u \xrightarrow[s \to \infty]{} \gamma_1 \end{array} \right\} \Big/ \mathbb{R}$$

$\mathcal{M}(\gamma_0, \gamma_1)$ smooth of dimension $\deg(\gamma_0) - \deg(\gamma_1) - 1$

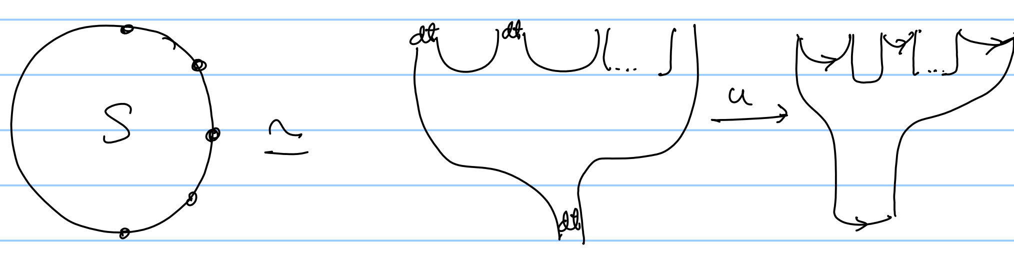



$$\mu^1(\gamma_1) = \sum_{\gamma_0} \left( \# \mathcal{M}(\gamma_0, \gamma_1) \right) \gamma_0$$

Wrapped Floer cohomology $HW^*(L_0, L_1)$

Composition maps

$$\mu^d : CW(L_{d-1}, L_d) \otimes \cdots \otimes CW(L_0, L_1) \to CW(L_0, L_d)$$



$\tilde{H}: S \to \mathcal{H}(M)$, $\beta \in \Omega^1(S)$, $J: S \to \mathcal{J}(M)$

$(du - X_{\tilde{H}} \otimes \beta)^{0,1} = 0$, $d\beta \leq 0$

use rescaled Hamiltonian chords ($\psi^\rho$).

$\rightsquigarrow$ well-defined composition maps $\mu^d$.

<u>Split generation</u>:

$\mathcal{A}$ $A_\infty$ category.

$\mathcal{A}$-mod, mod-$\mathcal{A}$, $\mathcal{A}$-$\mathcal{A}$ mod

( left right bi -modules )

<u>Yoneda embedding</u>:

$Y_L : \mathcal{A} \hookrightarrow \mathcal{A}\text{-mod}$

$Y_R : \mathcal{A} \to \text{mod } \mathcal{A}$

$Y_L(K)(X) = \hom_A(K, X)$

$Y_L(K)(X) = \hom_A(X, K)$

$HH_*(\mathcal{A}, \mathcal{B})$ $\leftarrow$ $\mathcal{A}$-$\mathcal{A}$ bimodule

<u>Def.</u> $W$ $A_\infty$-category, $\mathcal{B} \subset W$

$\mathcal{B}$ split generates $W$ if $\forall K \in Ob(W)$

$Y_L(K)$ admits a cohomologically

left invertible morphism into a

twisted complex of Yoneda modules of $\mathcal{B}$.

<u>Symplectic cohomology</u>:

$(M, \omega = d\lambda)$

$H$ admissible Hamiltonian

$F : S^1 \times M \longrightarrow \mathbb{R}$

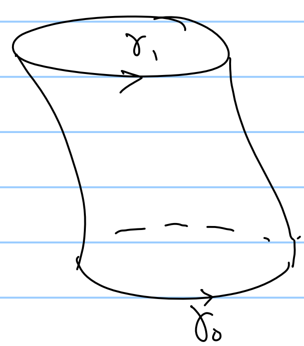
$G = H + F \rightsquigarrow X_G$

$\mathcal{O} = \{ \gamma : S^1 \to M \mid \dot{\gamma} = X_G \circ \gamma \}$

Symplectic chain complex:

$$SC(M) = \bigoplus_{\mathcal{O}} \mathbb{K} \cdot \gamma$$

$$\mathcal{M}(\gamma_0, \gamma_1) = \left\{ u : \mathbb{R} \times S^1 \longrightarrow M \;\middle|\; \begin{array}{l} (du - X_G \otimes dt)^{0,1} = 0 \\ u \xrightarrow{s \to \infty} \gamma_1 ,\; u \xrightarrow{s \to -\infty} \gamma_0 \end{array} \right\}$$




$\partial : SC^*(M) \circlearrowleft$

$$\partial(\gamma_1) = \sum_{\gamma_0} \left( \# \mathcal{M}(\gamma_0, \gamma_1) \right) \gamma_0$$

$$\rightsquigarrow SH^*(M) := H^*(SC^*(M), \partial)$$

Rem: Thm: $SH^*(T^*Q, d\lambda_{can}) \simeq H_*(\mathcal{L}Q)$

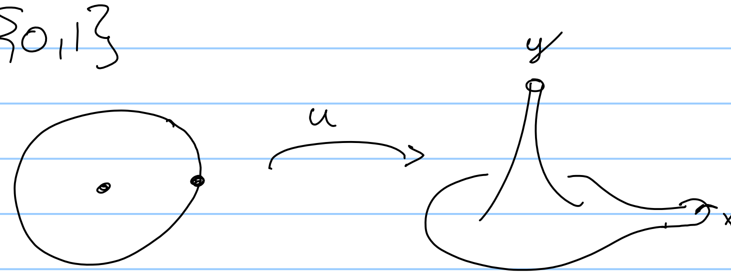
$\mathcal{L}Q$ : free loop space.

Closed open map:

Chain map $\mathcal{CO}: \underset{\substack{\cup \\ y}}{SC^*(M)} \longrightarrow \underset{\substack{\cup \\ x}}{CW^*(K,K)}$

$\forall K \in Ob(W)$

$S = D \setminus \{0,1\}$




$\mathcal{M}(x,y)$

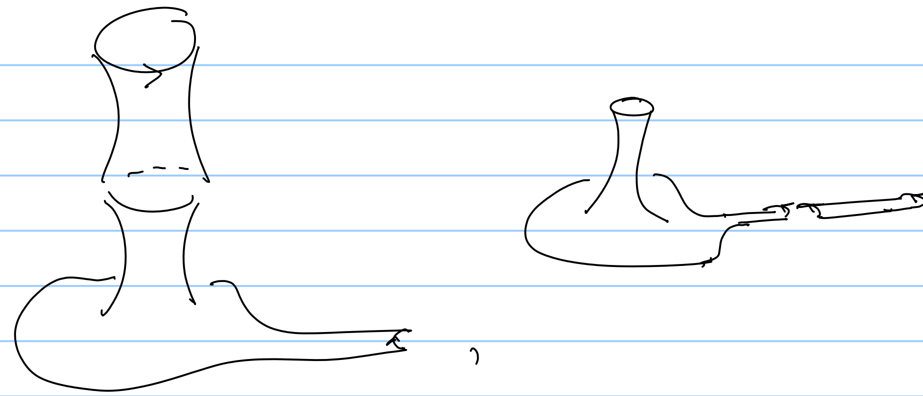
$$\mathcal{CO}(y) = \sum_{x} (\# \mathcal{M}(x,y))\, x$$

s.t. moduli space has dim $=0$

Prop $\mathcal{CO}$ is a chain map,

$$\mathcal{CO} \circ \partial = \mu^1 \circ \mathcal{CO}$$

Proof: Gromov compactness of $\mathcal{M}$

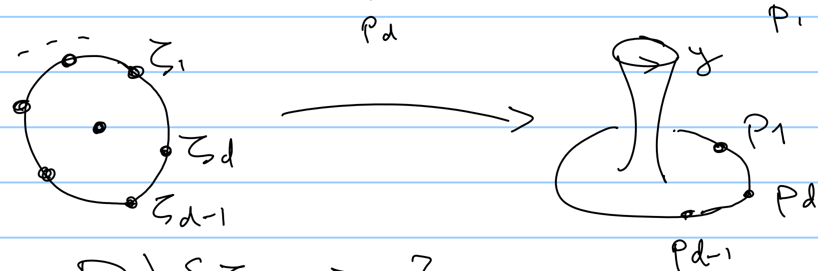


## Open-closed map

Chain map

$$OC : CC_*(\mathcal{B},\mathcal{B}) \longrightarrow SC^*(M)$$

$$OC^d : CW^*(\underset{\substack{\Downarrow \\ p_d}}{L_{d-1}, L_d}) \otimes \cdots \otimes CW(\underset{\substack{\Downarrow \\ p_1}}{L_0, L_1}) \to SC^*(M)$$

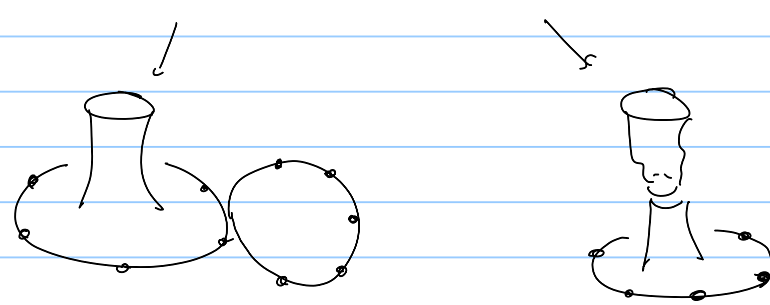



$S = D \setminus \{\zeta_0, \dots, \zeta_d, 0\}$

$\mathcal{M}(p_d, \dots, p_1, y)$

$$OC^d(p_d, \dots, p_1) = (-1)^* \sum \left(\# \mathcal{M}(p_d, \dots, p_1, y)\right) y$$

**Prop** $\quad OC \circ b = \partial \circ OC$



Proof (Abouzaid)

$K \in ob(W)$, $\quad \mathcal{B} \subset W$

Assumption $HH_*(\mathcal{B}, \mathcal{B}) \xrightarrow{H^*(OC)} SH^*(M)$

hits identity

$Y_L^K \in W\text{-mod} \qquad Y_R^K \in \text{mod-}W$

$W\text{-mod} \longrightarrow \mathcal{B}\text{-mod}, \qquad \text{mod-}W \longrightarrow \text{mod-}\mathcal{B}$

Image $\qquad$ Image

Denote images: $YB_L^K \qquad YB_R^K$

$$H^*(YB_R^K \otimes_{\mathcal{B}} YB_L^K) \xrightarrow{H^*(\mu)} HW^*(K,K)$$

Lemma If $Id \in HW^*(K,K)$ lies in the image of $H^*(\mu)$, then $\mathcal{B}$ split-generates $K$.

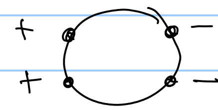
Prop.

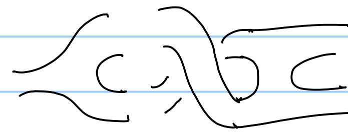
$$\begin{tikzcd} HH_*(\mathcal{B}, \mathcal{B}) \arrow[r, "HH_*(\Delta)"] \arrow[d, "H^*(OC)"'] \arrow[dr, phantom, "\text{commutes}"] & H^*(YB_R^K \otimes YB_L^K) \arrow[d, "H^*(\mu)"] \\ SH^*(M) \arrow[r, "H^*(CO)"'] & HW^*(K,K) \end{tikzcd}$$

$\underbrace{\phantom{H^*(CO)}}_{\text{unital}}$

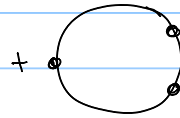
## Simplest case

$HH_*(\Delta)$:

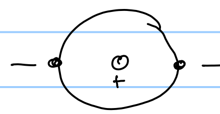




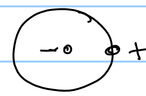
$H^*(\mu)$

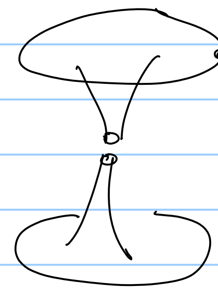


$H^*(\mathcal{OC})$

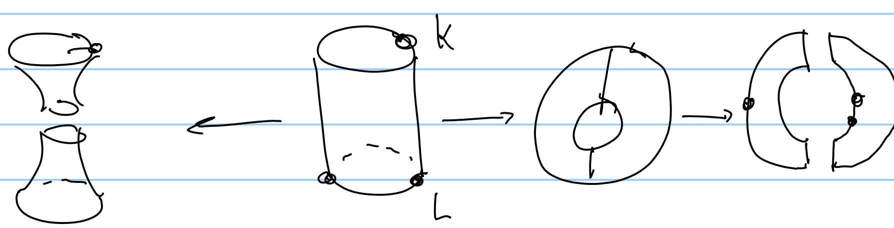


$H^*(\mathcal{CO})$





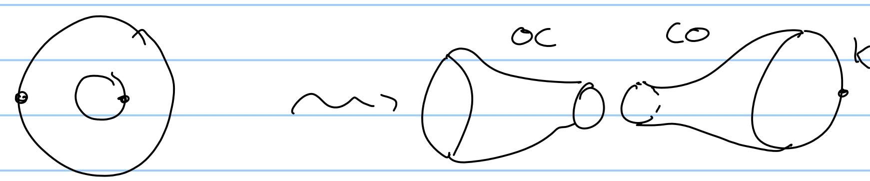
$h$ homotopy $h: CC_*(\mathcal{B},\mathcal{B}) \longrightarrow CW^*(K,K)$

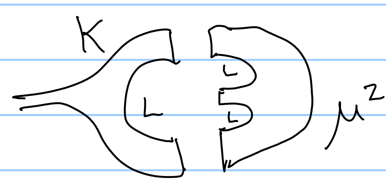


Simpler example (Nick)

$$\begin{tikzcd} HW^*(L,L) \arrow[r, "OC"] \arrow[d] & SH^* \arrow[d, "CO"] \\ HW^*(K,L) \otimes HW^*(L,K) \arrow[r] & HW^*(K,K) \end{tikzcd}$$

$$K \underset{q}{\overset{p}{\rightleftarrows}} L$$




Can add more marked points on L boundary to get higher story.