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Note Title 6/10/2015

Fukaya Category as A_∞ category

$(M, \omega = d\lambda)$ exact symplectic manifold

$L_0, \dots, L_d \subset M$ exact compact Lagrangians

(i.e. $\lambda|_{L_i} = df_i$, $f_i : L_i \to \mathbb{R}$)

Assume $L_i \pitchfork L_{i+1}$ for all i (mod $d+1$)

Floer complex

$$F(L_i, L_{i+1}) = \mathbb{K}\langle L_i \cap L_{i+1} \rangle$$

Composition maps

$J \in \mathcal{J}(M) := \{\omega\text{-comp almost complex structures}\}$

$\beta \in \pi_2(M, L_0 \cup \dots \cup L_d)$,

$\operatorname{ind}(\beta) =$ Maslov index of β

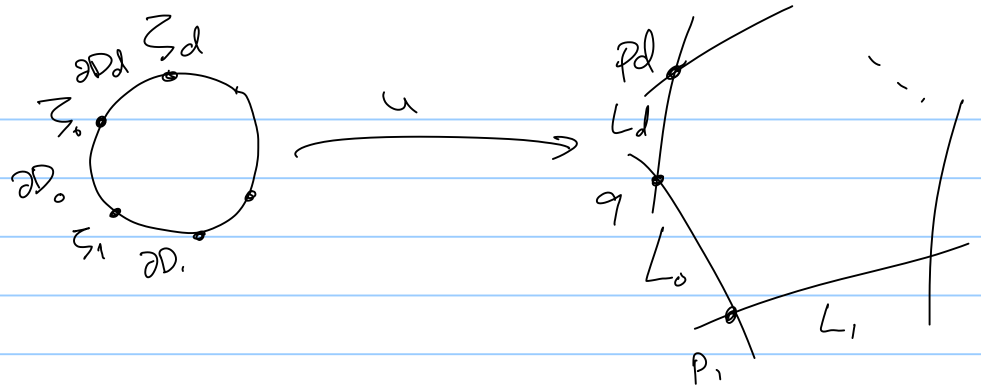
$D := \{|z| \le 1\} \subset \mathbb{C}$

For $p_i \in CF(L_i, L_{i+1})$

$q \in CF(L_0, L_d)$ we define

$\mathcal{M}(p_d, \dots, p_1, q, \beta, J) =$

$$= \left\{ \begin{array}{l|l} (\zeta_0, \dots, \zeta_d) \text{ ordered pts on } \partial D & (du)^{0,1} = du + J \circ du \circ j = 0 \\ u : D \setminus \{\zeta_0, \dots, \zeta_d\} \to M & u(\partial D_i) \subset L_i \\ & u \xrightarrow[z \to \zeta_i]{} p_i, q \\ & [u] = \beta \end{array} \right\} \Big/ \sim$$



$\sim$: action of $PSL(2,\mathbb{R}) \cong \{\text{hol automorphisms of } D\}$

$$((\zeta_0,\dots,\zeta_d),u) \sim ((\varphi(\zeta_0),\dots,\varphi(\zeta_d)), u\circ\varphi^{-1})$$

Under genericity assumptions (transversality) and further conditions on the L_i, it follows that $\mathcal{M}(p_d,\dots,p_1,q,\beta,J)$ is a smooth manifold of

$$\dim = d-2+\operatorname{ind}(\beta)$$

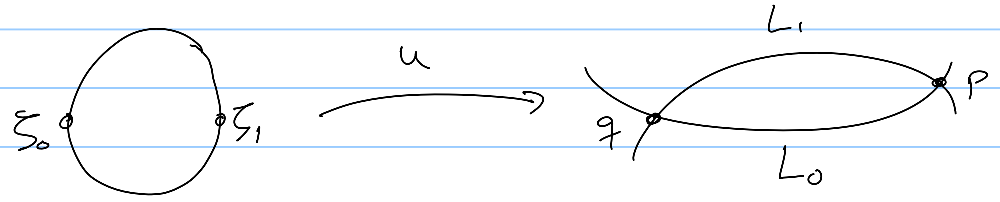
<u>Def:</u> Composition maps for $d\geq 1$ ($\mathbb{K}$-linear)

$$\mu^d : CF(L_{d-1},L_d)\otimes\dots\otimes CF(L_0,L_1)\to CF(L_0,L_d)$$

$$\mu^d(p_d,\dots,p_0) := \sum_{\substack{q\in CF(L_0,L_d)\\ \beta:\operatorname{ind}(\beta)=2-d}} (\#\mathcal{M}(p_d,\dots,p_1,q,\beta,J))\,q$$

Examples $d=1$

$$\mu^1 : CF(L_0, L_1) \longrightarrow CF(L_0, L_1)$$



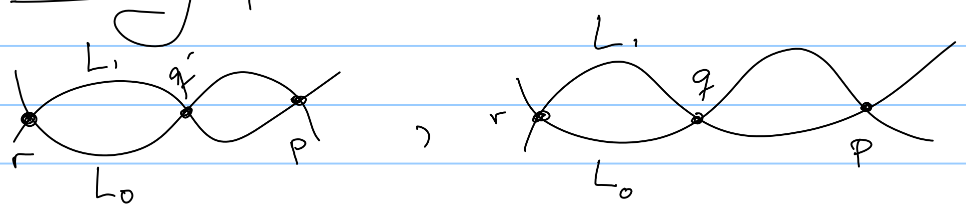
Prop: $\mu^1 \circ \mu^1 = 0$

Idea: Consider Gromov compactification of $\mathcal{M}(p, r, \beta, J)$ for $\mathrm{ind}(\beta) = 2$

Boundary points: broken strips, disk bubbles, sphere bubbles

Exactness assumptions $\longrightarrow \nexists$ non-trivial J-hol disks and spheres

Boundary points:



Gluing theorem: all broken strips are boundary points $\Longrightarrow$ claim ☒

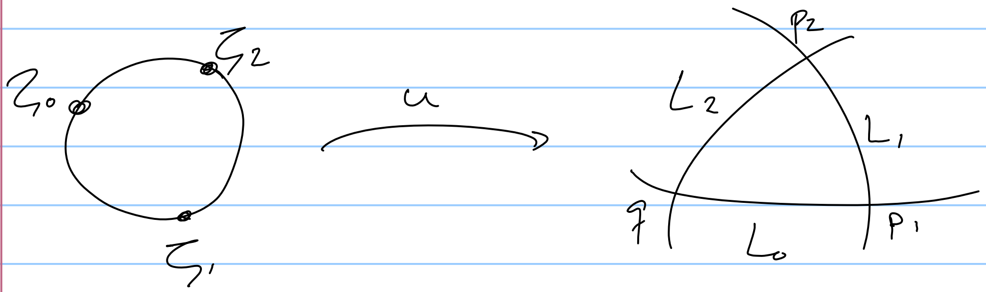
Define: $HF(L_0, L_1) = H(CF(L_0, L_1), \mu^1)$

Floer cohomology of L_0 & L_1

Prop. L_1 Ham isotopic to L_0

$\longrightarrow$ $HF(L_0, L_1) \cong H^*(L_0)$

2) $d = 2$



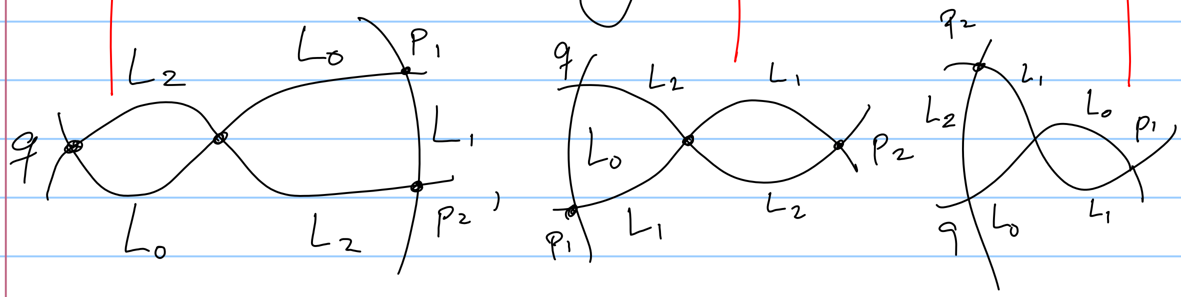
Prop: μ^2 satisfies the Leibniz rule.

$$\mu^1 \circ \mu^2(p_2, p_1) = \pm \mu^2(\mu^1(p_2), p_1) \pm \mu^2(p_2, \mu^1(p_1))$$

Idea: Gromov compactification of

$\mathcal{M}(p_2, p_1, q, \beta, J)$ for $\mathrm{ind}(\beta) = 1$

$\Rightarrow$ 3 possible boundary points



With (*) μ^2 induces multiplication

$$[\mu^2] : HF(L_2, L_1) \otimes HF(L_0, L_1) \longrightarrow HF(L_0, L_2)$$

Rem μ^2 associative up to homotopy given by μ^3

$\Rightarrow [\mu^2]$ is associative

Donaldson - Fukaya category

$DF(M)$ $\mathbb{K}$-linear category

- Objects: exact compact Lagrangians
- Morphisms: $HF(L_0, L_1)$ $\forall L_0, L_1$ objects
- Composition of morphisms: $[\mu^2]$.
- Identity morphisms: $e_i \in HF^*(L, L)$

Def. A_∞-category $\mathcal{A}$

- set of objects $Ob(\mathcal{A})$
- graded vector spaces

$$hom^*_{\mathcal{A}}(X_0, X_1), \quad X_0, X_1 \in Ob(\mathcal{A})$$

- composition maps

$$\mu^d : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \longrightarrow \hom^*_{\mathcal{A}}(X_0, X_d)[2-d]$$

that satisfy A_∞-equations:

$$0 = \sum_{m=1}^{d} \sum_{n=0}^{d-m} (-1)^{*} \mu_{\mathcal{A}}^{d-m+1}(a_d, \ldots, a_{n+m+1}, \mu_{\mathcal{A}}^{m}(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots, a_1)$$

$$* = |a_1| + \cdots + |a_n| - n$$

<u>Fukaya category:</u>

$\mathcal{F}(M)$ $\quad\mathbb{K}$-linear A_∞-category

- Objects: Lagrangians $L \subset M$
(exact, compact, oriented, spin
$\mu_L = 0$, grading)

- $\hom^*_{\mathcal{F}(M)}(L_0, L_1) = CF^*(L_0, L_1)$
$\forall L_0, L_1 \in Ob(\mathcal{F}(M))$

- Composition maps

$$\mu^d : CF^*(L_{d-1}, L_d) \otimes \cdots \otimes CF^*(L_0, L_1) \to CF^*(L_d, L_0)$$

Gromov compactification $\Longrightarrow$ A_∞-eqn's.

Monotone Fukaya category

(M, ω) monotone:

$$[\omega] = 2\tau c_1, \quad \tau > 0$$

$L \subset M$ Lagrangian:

1) $\pi_1(L) \to \pi_1(M)$ trivial

2) $[\omega] : H_2(M, L) \to \mathbb{R}$

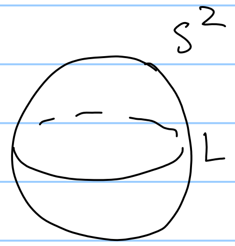
$\mu : H_2(M, L) \to \mathbb{Z}$

$[\omega] = \tau \mu.$

3) L oriented, spin

mon Lagrangians bound J-hol disks

Ex.



L monotone $\longleftrightarrow$ splits S^2 equally

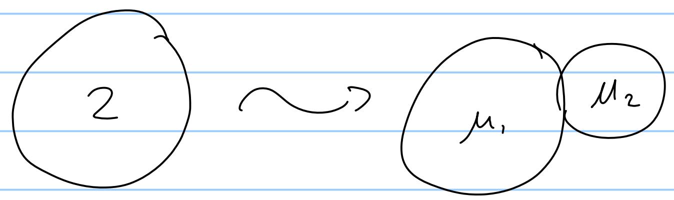
L oriented $\longrightarrow$ $\mu_L \geq 2.$

$\exists J \in \mathcal{J}(M), \quad \beta \in \pi_2(M, L), \ \mu(\beta) = 2.$

Lemma: $\mathcal{M}(\beta, J) := \left\{ J\text{hol disks on } L \text{ with one marked point} \right\}$?

is a smooth oriented compact manifold

of $\dim = n = \dim L$.



$ev : \mathcal{M}(\beta, J) \longrightarrow L$,

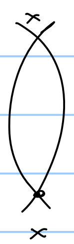
$\nu(\beta) =$ degree of ev

$W(L) := \sum_{\beta} \nu(\beta) \in \mathbb{Z}$

L_0, L_1 monotone: $L_0 \pitchfork L_1$

$\rightsquigarrow \mu^1$

$\mu^1 \circ \mu^1(x) = (W(L_0) - W(L_1)) \cdot x$



$\Rightarrow \forall\, w \in \mathbb{Z}(\mathbb{C})$

get $F(M)_w$ is Fukaya category

with objects $L \subset M$ with

$$W(L) = w.$$