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Note Title 11/06/2015

## Matrix factorizations

$R$ – commutative ring

$\cup$

$W$

Spec $R$ – smooth variety

Spec $R/W$ – singular

$MF(R,W)$ – $\mathbb{Z}_2$ graded dg cats

$MF(R,W) \sim$ B-side of mirror symmetry

$Coh^b(\mathrm{Spec}\, R/W)$

$$DB := \prod [MF(R, W-a)]$$

fiber over a is singular

Definition $MF(R,W)$

Obj: $(X,d)$

$X$ – 2 graded free $R$-module

$X = X_0 \oplus X_1$

$d \circlearrowright X$ s.t. $d^2 = W\,\mathrm{Id}$

odd: $X_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} X_1 \qquad d = p_0 + p_1$

__Mor__ $MF(X, X')$

$\mathbb{Z}_2$ graded module of morphisms

$f \in MF(X, X')$

$$df = d_{X'} \circ f - (-1)^{|f|} f \circ d_X .$$

__Claim:__ $d^2 = 0$.

__Def.__ $[MF(R,W)]$ - an additive category with the same objects and morphisms given by

$$H^0(MF(X, X'))$$

__Claim:__ $[MF(R,W)]$ is triangulated

__Example__ $R = \mathbb{C}[x]$ $\quad W = x^n$

$$\begin{tikzcd}
R \arrow[r, shift left, "x^{n-k}"] \arrow[d, phantom, "\parallel"] \arrow[dd, bend right=60, "a"'] & R \arrow[l, shift left, "x^k"] \arrow[d, phantom, "\shortparallel"] \arrow[dd, bend left=60, "b"] & \in MF(R,W) \\
X_0 & X_1 & \\
R \arrow[r, shift left, "x^{n-l}"] & R \arrow[l, shift left, "x^l"] &
\end{tikzcd}$$

__Exercise:__ Compute morphisms between such objects in $[MF(R,W)]$

Answer $R/(x^{\min(k,n-\ell)})$

Objects of $MF(R,W)$ as sheaves

$D_{sg}(W^{-1}(0)) \rightsquigarrow$ Orlov

$X$ - nice variety, not smooth

$D^b(\mathrm{Coh}\, X) \supset \mathrm{Perf}$

objects are isomorphic to bdd complexes of locally free, finite rank sheaves on $X$

Perf is a full triangulated subcategory

$D_{sg}(X) := D^b(\mathrm{Coh})/\mathrm{Perf}$

triangulated

$MF(R,W)$

$$\begin{array}{ccc} \mathrm{Spec}\, R & \xrightarrow{W} & \mathbb{A}^2 \\ \cup & & \\ W^{-1}(0) & & \end{array}$$

## Theorem (Orlov)

$$[MF(R, W)] \simeq D_{sg}(W^{-1}(0))$$

exact equivalence

## Map:

$$0 \longrightarrow X_1 \xrightarrow{p_1} X_0 \longrightarrow \operatorname{Coker} p_1 \longrightarrow 0$$

sheaf on spec $R$

$p_1 \circ p_0 = W \cdot id \rightsquigarrow W \cdot \operatorname{Coker} p_1 = 0$

sheaf on spec $R/W$

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$R = \mathbb{C}[x_1, \dots, x_n]$

Consider skyscraper sheaf at the origin in $\mathbb{A}^n$.

We will compute a corresponding object in $MF(R,W)$ for $W \in R$ with singularity at the origin.

$$\begin{cases} W \in (x_1, \dots, x_n) \\ dW = 0 \text{ at } 0 \end{cases}$$

Skyscraper sheaf $\leftrightarrow R/(x_1, \dots, x_n)$

$$\overbrace{\Lambda^n R \to \cdots \to \Lambda^i R^n \cdots \xrightarrow{d_0^1} R^n \to R}^{B} \to R/(x_1,\dots,x_n)$$

$R^n$ has a basis $\theta_1,\dots,\theta_n$

$d(\theta_{i_1}\wedge\dots\wedge\theta_{i_k}) =$

$$= \sum_{j=1}^{k} (-1)^{j+1} x_j\, \theta_{i_1}\wedge\dots\wedge\hat{\theta}_{i_j}\wedge\dots\wedge\theta_{i_k}$$

$$\begin{tikzcd}
B_n \arrow[r] \arrow[d, "W"] & B_{n-1} \arrow[r] \arrow[d, "W"] & \cdots \arrow[r] & B_1 \arrow[r] \arrow[d, "W"'] & B_0 \arrow[r] \arrow[d, "W"] \arrow[dl, "h"'] & 0 \\
B_n \arrow[r] & B_{n-1} \arrow[r] & \cdots \arrow[r] & B_1 \arrow[r] & B_0 \arrow[r] & 0
\end{tikzcd}$$

$W = \sum W_i x_i$ (since $W \in (x_1,\dots,x_n)$)

$h$ is given by $\wedge$ with $\sum w_i \theta_i$.

$$\bigoplus_{i=0}^{W} B_i \qquad (d_0+d_1)^2 = W$$

$(B, d_0+d_1) \in MF(R,W)$

$R[\theta_1,\dots,\theta_n] = B$

**Theorem** (Dyckerhoff) If $W$ has isolated sing at $0$, $(B, d_0 + d_1)$ generates the triangulated category $[MF^\infty(R,W)]$ under shifts and products

Consider

$$\mathrm{End}(B, d_0 + d_1) \simeq \underset{A}{\underset{\cup}{R[\theta_1, \dots, \theta_n, \tfrac{\partial}{\partial\theta_1}, \dots, \tfrac{\partial}{\partial\theta_n}]}}$$

differential is bracket

$$[d_1 + d_0, \quad ]$$

$$d_1 = \sum_{i=1}^{n} w_i \theta_i$$

$$d_0 = \sum_{i=1}^{n} \frac{\partial}{\partial\theta_i} \cdot X_i$$

We want to compute the $A_\infty$ structure on $H(A) = \mathbb{C}[\partial_1, \dots, \partial_n]$

given by

homological perturbation lemma.

Assuming $W$ is of degree $\geq 3$.

We can recover $W$ from the $A_\infty$-structure on $H(A)$.

$$CC^*(H(A)) \xrightarrow{\Phi} \mathbb{R}\left[\frac{\partial}{\partial\theta_1}, \ldots, \frac{\partial}{\partial\theta_n}\right]$$

Maurer-Cartan ell $\qquad\qquad W_{\underline{i}} \dfrac{\partial}{\partial\theta_{i_1}} \cdots \dfrac{\partial}{\partial\theta_{i_n}}$

$\updownarrow$

$A_\infty$-str on $H(A)$ $\qquad\qquad W = \sum_{\underline{i}} W_{\underline{i}}\, x^{\underline{i}}$

$\underline{i} = (i_1, \ldots, i_k)$.