Note Title 11/06/2015

Brezner II

$$D^b(\mathbb{P}^n) = \langle \mathcal{O}(-n), \dots, \mathcal{O} \rangle$$

1) The complex $R\Gamma(\mathbb{P}^n, F^\bullet \otimes \Omega^k(k))$ does not have a triv. diff.

2) Y a proj var. $F \in Coh(Y)$
$\exists m \in \mathbb{Z}$

$$\bigoplus_{fin} \mathcal{O}(m) \longrightarrow F \to 0$$

3) $\mathrm{Coh}\, Y \hookrightarrow \mathrm{QCoh}\, Y$ has enough inj.'s

$$\Downarrow$$

$$D^b(\mathrm{Coh}\, Y) \xrightarrow{F} D^b(\mathrm{QCoh}\, Y)$$

if Y is a var, F is full + faith w/ ess image consisting of complexes w/ coherent cohomology.

Thm. $i: Y \hookrightarrow \mathbb{P}^n$ closed emb. smooth and $F_1, \dots, F_n$ split generate $D^b(\mathbb{P}^n)$ then $Li^* F_i$ split generate $D^b(Y)$.

Def. $F^\circ \xrightarrow{e} F^\circ$ in $D^b(Y)$ s.t. $e^2 = e$ is called an idempotent.
It is split if

$$F \xrightarrow{f} \mathcal{G}^\circ \xrightarrow{g} F$$

s.t. $g \circ f = e$, $f \circ g = 1_{\mathcal{G}^\circ}$

Claim: If Y is a proj. smooth variety then every idem. in $D^b(Y)$ is split.

Let T be a tri. cat w/infinite direct sums.
Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to$ be a directed system in T

diag: $\bigoplus_i X_i \longrightarrow \bigoplus_i X_i$

given by (next page)

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & X_2 \\ \oplus & & \oplus \\ X_2 & \xrightarrow{f_2} & X_3 \\ & \vdots & \end{array}$$

$$\bigoplus_i X_i \xrightarrow{1-\mathrm{diag}} \bigoplus_i X_i \longrightarrow \underset{\underset{\displaystyle \operatorname{hocolim}_{\rightarrow} X_i}{\parallel}}{Y} \longrightarrow \left(\bigoplus X_i\right)[1]$$

$X \xrightarrow{1} X \xrightarrow{1} X \longrightarrow \qquad \operatorname{hocolim}(X, 1) = X$

$X \xrightarrow{0} X \xrightarrow{0} X \longrightarrow \qquad \operatorname{hocolim}(X, 0) = 0$

Let $X \xrightarrow{e} X$ be an idempotent

$$\begin{tikzcd} X \arrow[r, "e"] & X \arrow[r, "e"] & X \arrow[r] \arrow[rr, bend left, "f"] & \cdots & \operatorname{hocolim}(X, e) = Y \end{tikzcd}$$

$\rightsquigarrow$

$X \xrightarrow{1-e} X \xrightarrow{1-e} X \longrightarrow \cdots \qquad \operatorname{hocolim}(X, 1-e) = Z$

$$\begin{tikzcd}[column sep=large] X \oplus X \arrow[r, "\begin{pmatrix} e & 0 \\ 0 & 1-e \end{pmatrix}"] \arrow[d, "\begin{pmatrix} e & 1-e \\ 1-e & e \end{pmatrix}"'] & X \oplus X \arrow[r, "\begin{pmatrix} e & 0 \\ 0 & 1-e \end{pmatrix}"] & X \oplus X \arrow[r] & \cdots \arrow[r] & Y \oplus Z \arrow[d, "(g\ g')"] \\ X \oplus X \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}"] & X \oplus X \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}"] & \cdots & & X \end{tikzcd}$$

$(g, g')(y, z) = g(y) + g'(z)$

$$\begin{pmatrix} e & 1-e \\ 1-e & e \end{pmatrix}^2 = \mathrm{Id}$$

$\Rightarrow X \simeq Y \oplus Z$

$\Rightarrow X \xrightarrow{f} Y \xrightarrow{g} X \qquad g \circ f = e$

$\qquad\qquad\qquad\qquad f \circ g = 1_Y$

$$\left(\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "e"'] & Y \arrow[d, "g"] \\ X \arrow[r, "\mathrm{id}"] & X \end{tikzcd} \right)$$

$F^\bullet \xrightarrow{e} F^\bullet$ an idemp. in $D^b(\mathrm{Coh}\, Y)$

$\Rightarrow$ e splits in $D^b(\mathrm{QCoh}\, Y)$

$$\mathcal{G}^\bullet \xrightarrow{g} F^\bullet \xrightarrow{f} \mathcal{G}^\bullet$$

$$\begin{tikzcd} H^i(\mathcal{G}^\bullet) \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & H^i(F^\bullet) \arrow[r] & H^i(\mathcal{G}^\bullet) \end{tikzcd}$$

$\Rightarrow$ $H^i(\mathcal{G}^\bullet)$ is coherent

$\Rightarrow$ $\mathcal{G}^\bullet \in D^b(\mathrm{Coh})$

$i : \underset{\text{smooth}}{Y} \hookrightarrow \mathbb{P}^n$ closed emb.

Suppose $F_1, \dots, F_m$ split gen $D^b(\mathbb{P}^n)$

$$\mathcal{O}_Y(m) = i^*\mathcal{O}_{\mathbb{P}^n}(m) = Li^*\mathcal{O}_{\mathbb{P}^n}(m)$$

So the $E_i = Li^* F_i$ split-gen $\mathcal{O}_Y(m)$

Let $F \in \mathrm{Coh}(Y)$

$$\mathcal{O}_Y^{r_2}(m_2) \longrightarrow \mathcal{O}_Y^{r_1}(m_1) \longrightarrow F \longrightarrow 0$$

$$0 \to \mathcal{E} \to \mathcal{O}_Y^{r_\ell}(m_\ell) \cdots$$

Set $\mathcal{G}^\bullet = \{0 \to \mathcal{O}_Y^{r_\ell}(m_\ell) \to \cdots \to \mathcal{O}_Y^{r_1}(m_1) \to 0\}$

$\Longrightarrow \quad \mathcal{G}^\bullet \to \mathcal{F} \to \mathcal{E}[\ell] \to \mathcal{G}^\bullet[1]$

exact trd

in $D^b(Y)$

$$\mathrm{Hom}_{D^b(Y)}(\mathcal{F}, \mathcal{E}[\ell]) = \mathrm{Ext}^\ell(\mathcal{F}, \mathcal{E}) = 0$$

Splitting lemma

$$\Longrightarrow \mathcal{G}^\bullet = \mathcal{F} \oplus \mathcal{E}[\ell-1]$$

So $\mathcal{F}^\bullet$ is split generated

by $\mathcal{O}_Y(m)$.