Brezner I

Derived cat of an ab cat.

($QCoh\, X$, $Coh\, X$, R-mod)

$$A \rightsquigarrow C(A) \xrightarrow{H^i} A$$

cat of complexes

$\rightsquigarrow K(A)$ the homotopy cat.

$$Ob\, K(A) = Ob\, C(A)$$

$$Hom(X^\bullet, Y^\bullet) = Hom_{C(A)}(X^\bullet, Y^\bullet)/\sim$$

$\sim$ = homotopy

$\Longrightarrow D(A)$ invert $\underbrace{q\text{-iso}}_{\text{iso on cohomology}}$ $\longrightarrow$

$$\begin{tikzcd} & Z^\bullet \arrow[dl, "q"'] \arrow[dr, "f"] & \\ X \arrow[rr, dashed, "f\circ q^{-1}"'] & & Y \end{tikzcd} \Bigg\}\ \text{general morphism}$$

$$\begin{tikzcd} K(A) \arrow[r, "F"] \arrow[d, "P_A"'] & D \\ D(A) \arrow[ur, dashed, "\exists!\, G"'] & \end{tikzcd}$$

any F taking q-iso to iso.

"localization of category at q-iso"

$D(A)$ is an additive category.

w/a triangulated structure

$$[1] : D(A) \longrightarrow D(A)$$

$$(X^\bullet[1])^n = X^{n+1}$$

$$d^n_{X[1]} = -d^{n+1}_X$$

A triangle

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z \xrightarrow{h} X^\bullet[1]$$

$$\begin{tikzcd} X \arrow[rr] & & Y \arrow[dl] \\ & Z \arrow[ul, "{[1]}"] & \end{tikzcd}$$

<u>Mapping cone</u>:

$$X^\bullet \xrightarrow{f} Y^\bullet$$

$$C_f^\bullet : \quad C_f^n = Y^n \oplus X^{n+1}$$

$$\begin{tikzcd} Y^n \arrow[d, "d_Y"'] & \oplus & X^{n+1} \arrow[d, "-d_X"] \arrow[dll, "f"'] \\ Y^{n+1} & \oplus & X^{n+2} \end{tikzcd} \qquad d_{C_f} = \begin{pmatrix} d_Y & f \\ 0 & -d_X \end{pmatrix}$$

Declare tri's of the form

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{\alpha} C_f^\bullet \xrightarrow{\beta} X[1]$$

exact

Then

$0 \to X \to Y \to Z \to 0$ SES

$\rightsquigarrow$

$X \to Y \to Z \to X[1]$ exact tri in $D(\mathcal{A})$

$\rightsquigarrow$

LES of cohomology

$H^i(Z) \to H^{i+1}(Y)$

$D^b(A)$ the bounded derived cat.

the full subcategory of $D(A)$ consisting of complexes with bounded cohomology (all $H^i = 0$ for $i \gg 0$, $i \ll 0$)

Let $\mathcal{A}$ an ab cat w/ enough injectives

$0 \to X \to Y \to Z \to 0$ SES in $\mathcal{A}$

(Any object can be embedded in injective.

Injective:
$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[dl, dashed] \\ & I & \end{tikzcd}$$
)

$$\begin{tikzcd} & 0 \arrow[r] & Z \arrow[r] & 0 \\ 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "id"] & Y \arrow[u, "g"] \arrow[r] & 0 \\ 0 \arrow[r] & X \arrow[r] & 0 & \end{tikzcd}$$

$\Big\}$ q-iso

$$\wr$$

$$Z \longrightarrow X[1].$$

$$\operatorname{Ext}^1_{\mathcal{A}}(Z,X) \xrightarrow{\sim} \operatorname{Hom}_{D^b(\mathcal{A})}(Z, X[1])$$

$$\operatorname{Ext}^n_{\mathcal{A}}(Z,X) = \operatorname{Hom}_{D^b(\mathcal{A})}(Z, X[n])$$

$R\operatorname{Hom}(Z^\bullet, X^\bullet)$ cx s.t.

$$H^n(R\operatorname{Hom}(Z^\bullet, X^\bullet)) = \operatorname{Hom}_{D^b(\mathcal{A})}(Z^\bullet, X^\bullet[n])$$

Similarly if $\mathcal{A}$ has enough flat obj.

we have $X^\bullet \overset{L}{\otimes} Y^\bullet$ a complex (well-defined only in $D(\mathcal{A})$)

$$H^{-n}(X \overset{L}{\otimes} Y) = \mathrm{Tor}_n(X, Y)$$

$X, Y \in \mathcal{A}$.

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shift, form cone,

Thm. $D^b(\mathbb{P}^n)$ is gen by

$$\mathcal{O}_{\mathbb{P}}(-n), \dots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n}$$

Beĭlinson's (exceptional) collection

X, Y smooth projective varieties $/\mathbb{C}$

$$\begin{tikzcd} & X \times Y \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & Y \end{tikzcd}$$

$\Rightarrow p, q$ flat

$$M^\bullet \in D^b(X \times Y)$$

$$\Phi_M : D^b(Y) \longrightarrow D^b(X)$$

$$F^\bullet \longmapsto Rq_*(M^\bullet \overset{L}{\otimes} p^* F^\bullet)$$

$\mathcal{E}^\bullet \in D^b(X) \qquad \mathcal{F}^\bullet \in D^b(Y)$

$\mathcal{E}^\bullet \boxtimes \mathcal{F}^\bullet := q^*\mathcal{E}^\bullet \overset{L}{\otimes} p^*\mathcal{F}^\bullet$

$X = Y \implies \Delta \subset X \times X$

$$\Phi_{\mathcal{O}_\Delta} \simeq \mathrm{id}_{D^b(X)}$$

Proof: $\Phi_{\mathcal{O}_\Delta} : D^b(\mathbb{P}^n) \longrightarrow D^b(\mathbb{P}^n)$

Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}(1)^{n+1} \to \underset{\text{tan bundle}}{T_{\mathbb{P}^n}} \to 0$$

$$\left(\begin{array}{l} 0 \to \underset{\text{Tautological}}{\mathcal{O}(-1)} \to \mathcal{O}^{n+1} \to Q \to 0 \\ \mathrm{Hom}(\mathcal{O}(-1), -) \\ 0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to \underset{= T_{\mathbb{P}^n}}{\mathrm{Hom}(\mathcal{O}(-1), Q)} \to 0 \end{array}\right)$$

$\otimes\, \mathcal{O}(-1)$

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{n+1} \to T(-1) \to 0$$

$$H^0(\mathbb{P}^n, \mathcal{O}(-1)) = H^1(\mathbb{P}^n, \mathcal{O}(-1)) = 0$$

$$\implies H^0(\mathbb{P}^n, \mathcal{O}^{n+1}) = H^0(\mathbb{P}^n, T(-1)) \simeq \mathbb{C}^{n+1}$$

$$H^0(\mathbb{P}^n, \mathcal{O}(1)) \simeq \mathbb{C}^n$$

$$\mathcal{O}(1) \boxtimes T(-1) \quad \text{on} \quad \mathbb{P}^n \times \mathbb{P}^n$$

$$H^0(\mathbb{P}^n \times \mathbb{P}^n, \mathcal{O}(1) \boxtimes T(-1)) \simeq$$

$$\simeq H^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n, T(-1))$$

fix a basis $y_0, \dots, y_n$ of $H^0(\mathcal{O}(1))$

$\rightsquigarrow$ duals $y_0^\vee, \dots, y_n^\vee$

$$\longrightarrow s = \sum_{i=0}^{n} y_i \boxtimes y_i^\vee .$$

$p \in \mathbb{P}^n \times \mathbb{P}^n$ s.t. $s(p) = 0$

$$\Updownarrow$$

$p \in \Delta$

$$\Longrightarrow \quad
\begin{array}{ccc}
(\mathcal{O}(1) \boxtimes T(-1))^\vee & \xrightarrow{s} & \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \\
\wr\!\| & & \downarrow \\
\underbrace{\mathcal{O}(-1) \boxtimes \Omega(1)}_{\text{loc free of finite rank}} & & \mathcal{O}_\Delta \longrightarrow 0
\end{array}$$

Koszul complex

$$0 \longrightarrow \Lambda^n(\mathcal{O}(-1) \boxtimes \Omega(1)) \longrightarrow \cdots \longrightarrow \mathcal{O}(-1) \boxtimes \Omega(1) \longrightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

$$\Lambda^k(\mathcal{O}(-1) \boxtimes \Omega(1)) \simeq \underbrace{\mathcal{O}(-k) \boxtimes \Omega^k(k)}_{= L^k}$$

$$\Phi_{L^n}(F^\bullet) \longrightarrow \Phi_{L^{n-1}}(F^\bullet) \longrightarrow \Phi_{C_{n-1}}(F^\bullet)$$

$$\longrightarrow \Phi_{L^n}(F^\bullet)[1]$$

$$\vdots \qquad\qquad\qquad\qquad F^\bullet$$

$$\qquad\qquad\qquad\qquad\qquad \shortparallel$$

$$\Phi_{C_1}(F^\bullet) \longrightarrow \Phi_{\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}}(F^\bullet) \longrightarrow \Phi_{\mathcal{O}_\Delta}(F^\bullet)$$

$$\longrightarrow \Phi_{C_1}(F^\bullet)[1]$$

$\Rightarrow$ $F^\bullet$ is gen by

$$\{\Phi_{L^k}(F^\bullet)\}_{k=0}^n$$

$$\Phi_{L^k}(\mathcal{F}^\bullet) = Rq_*\left(p^*\mathcal{F}^\bullet \overset{L}{\otimes} q^*\mathcal{O}(-k) \overset{L}{\otimes} p^*\Omega^k(k)\right)$$

$$= Rq_*\left(q^*\mathcal{O}(-k) \overset{L}{\otimes} p^*(\mathcal{F}^\bullet \otimes \Omega^k(k))\right)$$

$$\simeq \mathcal{O}(-k) \otimes \underbrace{R\Gamma(\mathbb{P}^n, \mathcal{F}^\bullet \otimes \Omega^k(k))}_{\text{complex of fin dim vect spaces}}$$

$$\left(\Phi_{L^k}(\mathcal{F}^\bullet)\right)^n = \bigoplus_{\dim H^n(\mathcal{F}^\bullet \otimes \Omega^k(k))} \mathcal{O}(-k)$$

$\square$