

# Breznar I

Note Title

6/10/2015

Derived cat of an ab cat.

$$(\mathcal{Q}\text{Coh } X, \text{Coh } X, \mathbb{R}\text{-mod})$$

$$A \rightsquigarrow C(A) \xrightarrow{H^i} A$$

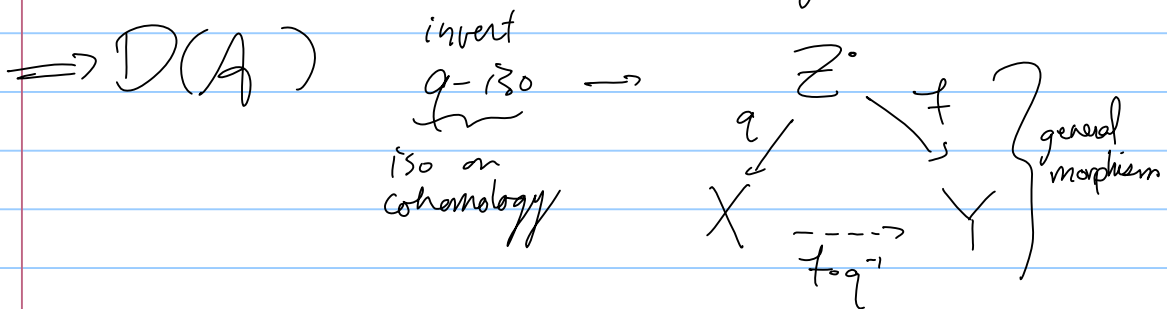
cat of complexes

$\rightsquigarrow K(A)$  the homotopy cat.

$$\text{Ob } K(A) = \text{Ob } C(A)$$

$$\text{Hom}(X^\bullet, Y^\bullet) = \text{Hom}_{C(A)}(X^\bullet, Y^\bullet) / \sim$$

$\sim = \text{homotopy}$



$$\begin{array}{ccc} K(A) & \xrightarrow{F} & D \\ \downarrow P_A & \dashrightarrow & \uparrow \exists! G \\ D(A) & & \end{array}$$

any  $F$  taking  $q$ -iso to iso.

"localization of category at  $q$ -iso"

$\mathcal{D}(A)$  is an additive category.

w/a triangulated structure

$$[1] : \mathcal{D}(A) \rightarrow \mathcal{D}(A)$$

$$(X^\bullet[1])^n = X^{n+1}$$

$$d_{X[1]}^n = -d_X^{n+1}$$

A triangle

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z \xrightarrow{h} X^\bullet[1]$$

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \nearrow [1] & \searrow \\ & Z & \end{array}$$

Mapping cone :

$$X^\bullet \xrightarrow{f} Y^\bullet$$

$$C_f^\bullet : C_f^n = Y^n \oplus X^{n+1}$$

$$\begin{array}{ccc} Y^n \oplus X^{n+1} & & \\ d_Y \downarrow \quad f \swarrow \quad \downarrow -d_X & & \\ Y^{n+1} \oplus X^{n+2} & & \end{array}$$

$$d_{C_f} = \begin{pmatrix} d_Y & f \\ 0 & -d_X \end{pmatrix}$$

Declare tri's of the form

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} C_{\neq}^0 \xrightarrow{\gamma} X[1]$$

exact

Then

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \quad \text{SES}$$

}

$$X \rightarrow Y \rightarrow Z \rightarrow X[1]$$

exact tri  
in  $D(A)$

}

LES of cohomology

$$H^i(Z) \rightarrow H^{i+1}(Y)$$

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$\underline{D}(A)$  the bounded derived cat.

The full subcategory of  $D(A)$   
consisting of complexes with

bounded cohomology (all  $H^i = 0$   
for  $i \gg 0$   
 $i \ll 0$ )

Let  $A$  an ab cat w/ enough injectives

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \quad \begin{array}{l} \text{SES} \\ \text{in } A \end{array}$$

(Any object can be embedded in injective.)

Injective: 
$$0 \rightarrow X \rightarrow Y$$

$$\begin{array}{ccc} & & \downarrow I \\ & & I \end{array}$$

$$\begin{array}{ccccccc} & & 0 & \rightarrow & Z & \rightarrow & 0 \\ & & & & \uparrow g & & \\ 0 & \rightarrow & X & \xrightarrow{f} & Y & \rightarrow & 0 \end{array} \left. \vphantom{\begin{array}{ccccccc} & & 0 & \rightarrow & Z & \rightarrow & 0 \\ & & & & \uparrow g & & \\ 0 & \rightarrow & X & \xrightarrow{f} & Y & \rightarrow & 0 \end{array}} \right\} \text{q-iso}$$

$$\begin{array}{ccc} & & \downarrow \text{id} \\ 0 & \rightarrow & X & \rightarrow & 0 \end{array}$$

$$Z \rightarrow X[1].$$

$$\text{Ext}_A^1(Z, X) \xrightarrow{\sim} \text{Hom}_{D^b(A)}(Z, X[1])$$

$$\text{Ext}_A^n(Z, X) = \text{Hom}_{D^b(A)}(Z, X[n])$$

$$\text{RHom}(Z^\bullet, X^\bullet) \text{ ex s.t.}$$

$$H^n(\text{RHom}(Z^\bullet, X^\bullet)) = \text{Hom}_{D^b(A)}(Z^\bullet, X^\bullet[n])$$

Similarly if  $A$  has enough flat obj.

we have  $X \overset{L}{\otimes} Y$  a complex (well-defined only in  $D(A)$ )

$$H^{-n}(X \overset{L}{\otimes} Y) = \text{Tor}_n^A(X, Y)$$

$$X, Y \in A.$$

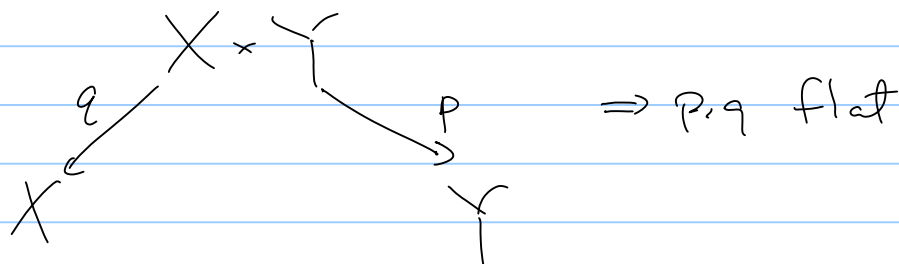
shift, form cone,

Thm.  $D^b(\mathbb{P}^n)$  is gen by

$$\mathcal{O}_{\mathbb{P}^n}(-n), \dots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n}$$

Beilinson's (exceptional) collection

$X, Y$  smooth projective varieties /  $\mathbb{C}$



$$M^\bullet \in D^b(X \times Y)$$

$$\mathbb{F}_\mu : D^b(Y) \longrightarrow D^b(X)$$

$$F^\bullet \longmapsto Rq_* (M^\bullet \overset{L}{\otimes} p^* F^\bullet)$$



$$H^0(\mathbb{P}^n, \mathcal{O}(1)) \cong \mathbb{C}^{n+1}$$

$$\mathcal{O}(1) \boxtimes \mathcal{T}(-1) \text{ on } \mathbb{P}^n \times \mathbb{P}^n$$

$$H^0(\mathbb{P}^n \times \mathbb{P}^n, \mathcal{O}(1) \boxtimes \mathcal{T}(-1)) =$$

$$\cong H^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{T}(-1))$$

fix a basis  $y_0, \dots, y_n$  of  $H^0(\mathcal{O}(1))$

$\rightsquigarrow$  dual  $y_0^\vee, \dots, y_n^\vee$

$$\rightarrow S = \sum_{i=0}^n y_i \boxtimes y_i^\vee$$

$$p \in \mathbb{P}^n \times \mathbb{P}^n \text{ s.t. } s(p) = 0$$

$$\updownarrow$$

$$p \in \Delta$$

$$\Rightarrow (\mathcal{O}(1) \boxtimes \mathcal{T}(-1))^\vee \xrightarrow{S} \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}$$

is

$$\mathcal{O}(-1) \boxtimes \mathcal{O}(1)$$

loc free of finite rank

$$\downarrow$$

$$\mathcal{O}_\Delta \rightarrow 0$$

# Koszul complex

$$0 \longrightarrow \Lambda^n(\mathcal{O}(-1) \boxtimes \Omega(1)) \longrightarrow \dots \longrightarrow \mathcal{O}(-1) \boxtimes \Omega(1) \longrightarrow \mathcal{O}_{\mathbb{P}^n / \mathbb{P}^n} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

$$\Lambda^k(\mathcal{O}(-1) \boxtimes \Omega(1)) \cong \mathcal{O}(-k) \boxtimes \Omega^k(k)$$

$\cong L^k$

$$\mathbb{F}_{L^n}(F^\bullet) \longrightarrow \mathbb{F}_{L^{n-1}}(F^\bullet) \longrightarrow \mathbb{F}_{\mathcal{O}_{\mathbb{P}^1}}(F^\bullet)$$

$$\longrightarrow \mathbb{F}_{L^1}(F^\bullet)[1]$$

⋮

$F^\bullet$   
⋮

$$\mathbb{F}_{\mathcal{O}_{\mathbb{P}^1}}(F^\bullet) \longrightarrow \mathbb{F}_{\mathcal{O}_{\mathbb{P}^1 / \mathbb{P}^n}}(F^\bullet) \longrightarrow \mathbb{F}_{\mathcal{O}_\Delta}(F^\bullet)$$

$$\longrightarrow \mathbb{F}_{\mathcal{O}_\Delta}(F^\bullet)[1]$$

$\Rightarrow F^\bullet$  is gen by

$$\left\{ \mathbb{F}_{L^k}(F^\bullet) \right\}_{k=0}^n$$



$$\mathbb{I}_{L^k}(F^\circ) = Rq_* \left( p^* F^\circ \otimes^L q^* \mathcal{O}(-k) \otimes^L p^* \Omega^k(k) \right)$$

$$= Rq_* \left( q^* \mathcal{O}(-k) \otimes^L p^* (F^\circ \otimes \Omega^k(k)) \right)$$

$$\cong \mathcal{O}(-k) \otimes \underbrace{R\Gamma(\mathbb{P}^1, F^\circ \otimes \Omega^k(k))}_{\text{complex of fin dim}}$$

vect spaces

$$\left( \mathbb{I}_{L^k}(F^\circ) \right)^n = \bigoplus_{\dim H^0(F^\circ \otimes \Omega^k(k))} \mathcal{O}(-k)$$

