Blaier II

From previous lecture

$$D^b(\mathrm{Fuk}(M)) = \mathrm{Tw}(\mathrm{Fuk}(M))$$

$$D^\pi(\mathrm{Fuk}(M)) = \Pi\,\mathrm{tw}(\mathrm{Fuk}(M))$$

Idempotents

Prelim discussion

Let A linear graded cat$/K$

Def ① $\rho \in \mathrm{Hom}_A(Y,Y)$, say ρ is idempotent if $\rho^2 = \rho$.

② The image of ρ to be $Z \in ob(A)$ with inclusion and retraction

$k \in \mathrm{Hom}(Z,Y)$, $r \in \mathrm{Hom}(Y,Z)$

$$Z \underset{r}{\overset{k}{\rightleftarrows}} Y \qquad rk = e_Z, \quad kr = \rho$$

Def. The split closure to be (B,F) such that B is split closed and

$$F : A \to B$$

is full + faithful embedding

such that each object in $\mathcal{B}$ is the image of some idempotent in $\mathcal{A}$.

split-closed = means that it contains the images of all idempotents.

<u>Def</u>: Let $\mathcal{A}$ be A_∞-cat. Then $\mathcal{A}$ is split-closed (A_∞) $\iff$ $H^0(\mathcal{A})$ split closed.

$\Pi(\mathrm{Tw}\,\mathcal{A})$

specific kind of split closure.

Hochschild homology

$\mathcal{A}$ A_∞-cat, K char $= 0$.

$$HH^*(\mathcal{A}, \mathcal{A}) = H(\hom^{\circ}_{\mathrm{fun}(\mathcal{A},\mathcal{A})}(id, id))$$

<u>What does it do?</u> classifies deformations

<u>Why is it awesome?</u>

commutative graded algebra
(shifted) Lie algebra

Gerstenhaber algebra

Two applications

— Classifying A_∞-str with fixed cohomology.

— HKR

I) Def. A graded unital algebra (B, m, η)

$$m : B \otimes B \longrightarrow B$$

$$\eta : k \longrightarrow B$$

s.t. (1) $m(m \otimes id) = m(id \otimes m)$

(2) $m(id \otimes \eta) = id = m(\eta \otimes id)$

Def. A coalgebra (C, Δ, ε) is a graded k vect sp. with

$$\Delta : C^\bullet \longrightarrow C^\bullet \otimes C^\bullet$$

$$C^\bullet \xrightarrow{\varepsilon} k$$

s.t.

$$\begin{tikzcd} C^\circ \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C^\circ \otimes C^\circ \arrow[d, "id \otimes \Delta"] \\ C^\circ \otimes C^\circ \arrow[r, "\Delta \otimes id"'] & C^\circ \otimes C^\circ \otimes C^\circ \end{tikzcd}$$

and

$$\begin{tikzcd} C^\circ \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C^\circ \otimes C^\circ \arrow[d, "\varepsilon \otimes id"] \\ C^\circ \otimes C^\circ \arrow[r, "id \otimes \varepsilon", "id \otimes \varepsilon"'] & C^\circ \end{tikzcd}$$ (counit)

<u>Def.</u> A morphism $f: (C, \Delta) \to (C', \Delta)$ of coalgebras is a linear map of graded v.s. satisfying

$$\Delta_{C'} f = (f \otimes f) \circ \Delta_C : C \to C' \otimes C'$$

<u>Example</u>: divided powers coalgebra

$$C^\cdot = k[x]$$

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}$$

$$\varepsilon(x^n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Bar construction

Let A be a graded k-v.s.

$$\Gamma(A[1]) = k \oplus A[1] \oplus (A \otimes A)[2] \oplus \cdots$$

$$\Delta : TA[1] \longrightarrow TA[1] \otimes TA[1]$$

$$a_\ell \otimes \cdots \otimes a_1 \longmapsto \sum_j (a_\ell \otimes \cdots \otimes a_{j+1}) \otimes$$
$$\otimes (a_j \otimes \cdots \otimes a_1)$$

$$\varepsilon : TA[1] \longrightarrow k$$

<u>Def</u>: a (deg $= d$) <u>coderivation</u>

is a linear map

$$\Gamma : C^{\bullet} \longrightarrow C^{\bullet + d}$$

such that

$$\Delta \circ \Gamma = (\Gamma \otimes id + id \otimes \Gamma) \circ \Delta$$

the space of coderivations is

$\mathrm{coder}^*(C)$.

<u>Def</u>. (1) A <u>dg-coalgebra</u>

is a coalgebra with $D_C \in \mathrm{coder}^1(C, C)$

such that $D_C^2 = 0$.

(2) A morphism of dg-coalg commutes with the differential.

Dictionary

A_∞-algebra	$TA[1]$
A_∞ str.	differential D, $D^0 = D\|_k = 0$
A_∞-homo	morphism of dg coalg
$\mathrm{Coder}(TA[1])[-1]$	Coderivations

Hochschild cochain space

$$CC^*(A,A) = \mathrm{Hom}(TA[1], A)$$

$$= \prod_{d\geq 0} \mathrm{Hom}^*(A^{\otimes d}, A)[-d]$$

Claim: $\mathrm{Coder}(TA[1]) \cong CC^*(A,A)[1]$

Proof:

$$\Pi \longmapsto \left(\begin{smallmatrix}\text{proj} \\ \text{to} \quad A[1]\end{smallmatrix}\right) \circ \Pi = (\gamma^d)_{d\geq 0}$$

$$\sum_{i,j} a_d \otimes \cdots \otimes \gamma^j(a_{i+j}, \ldots, a_{i+1}) \otimes \cdots \otimes a_1 \longleftarrow (\gamma^d)_{d\geq 0}$$

Def: Given

$\alpha, \beta \in CC^*(A, A)$

then <u>circle product</u> is

$$(\alpha \circ \beta)^d = \sum (-1)^* \alpha^{d-j+1}(a_d, \ldots, \beta(a_{i+j}, \ldots, a_{i+1}), \ldots, a_1).$$

$\underline{\mathrm{I}}$ Lie bracket (Gerstenhaber)

Prop: Let (C, Δ) be a graded coalg. Given $\theta, \phi \in \mathrm{Coder}(C)$ can define

$$[\theta, \phi] = \theta \circ \phi - (-1)^{|\theta||\phi|} \phi \circ \theta$$

Then $[\theta, \phi] \in \mathrm{Coder}(C)$ and $[-, -]$ makes $\mathrm{Coder}(C)$ into a graded Lie algebra.

Prop. Let (C, Δ) be a graded coalg. $\mu \in \mathrm{Coder}^1(C)$ s.t. $[\mu, \mu] = 0$. Then

$$d(\theta) = [\mu, \theta]$$

makes $\mathrm{Coder}(C)$ into dg-Lie alg.

Def. A non-unital A_∞ str. is a $\mu_A \in CC^2(A)$ s.t.

$$\mu^0 = 0, \quad \frac{1}{2}[\mu_A, \mu_A] = 0.$$

Def. A 1^{st} order deformation of the A_∞ alg A is a K_ε-linear ($K_\varepsilon = K[\varepsilon]/(\varepsilon^2)$) A_∞ str. on $A_\varepsilon = A \otimes_K K_\varepsilon$ which reduces to A if we set $\varepsilon = 0$. More concretely, introduce formal variable ε

$$\mu_{A_\varepsilon} = \mu_A + \varepsilon\beta, \quad \beta \in CC^2$$
$$\beta_0 = 0$$

$$0 = \frac{1}{2}[\mu_{A_\varepsilon}, \mu_{A_\varepsilon}]$$

$$= \frac{1}{2}[\mu_A, \mu_A] + \frac{1}{2}\varepsilon[\mu_A, \beta] + \frac{1}{2}\varepsilon[\beta, \mu_A] + \frac{1}{2}\cancelto{0}{\varepsilon^2}$$

$$\Longleftrightarrow \quad d(\beta) = [\mu_A, \beta] = 0.$$

<u>Def.</u> $G : V \longrightarrow V$ $\qquad V = TA[1] \otimes K_\varepsilon$

$$G = Id + \varepsilon f$$

$$G\mu_{A_\varepsilon} = \mu_{A'_\varepsilon} G$$

$$\begin{array}{ccc} \| & & \| \\ \mu_A + \varepsilon\beta & & \mu_A + \varepsilon\beta' \end{array}$$

$\Longleftrightarrow$ μ_{A_ε}, $\mu_{A'_\varepsilon}$ are <u>isomorphic</u>

CC^2 = Deformations

HH^2 = Deformation / iso

b/c $(id + \varepsilon f)(\mu_A + \varepsilon\beta) =$

$$= (\mu_A + \varepsilon\beta')(id + \varepsilon f)$$

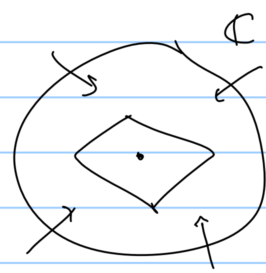
$$\Longrightarrow \quad \beta - \beta' = df.$$

III) Classification of A_∞ str with fixed cohomology

$\mathcal{U}(A)$ = space of A_∞-str with fixed cohomology A

$G(A)$ = formal diffeo

big picture



(A, m) graded algebra,

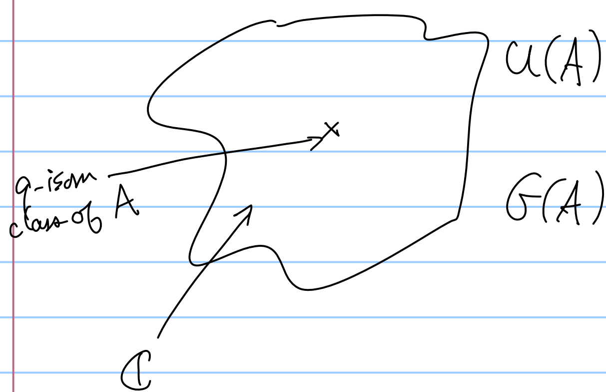
$\mathcal{U}(A) = \{ A_\infty\text{-str on } A \mid$

$$\mu^1 = 0$$
$$\mu^2 = (-1)^{|a_1|} m(a_2, a_1)$$

$G(A) =$

$\mathbb{C}^*$ action $\quad \varepsilon^* \mu^d = \varepsilon^{d-2} \mu^d$

big picture



$V^\bullet$ graded v. space

A a graded algebra

$m : V \otimes V \longrightarrow V$ on V

$\mathcal{A}$ A_∞ structure on $V^\bullet$:

$$\mu^d : \underbrace{V^\bullet \otimes \cdots \otimes V^\bullet}_{d} \longrightarrow V^\bullet[2-d]$$

$$\mathcal{U}(A) = \left\{ A_\infty \text{ structure on } \mathcal{A} \;\middle|\; \begin{array}{l} \mu^1 = 0 \\ \mu^2(a_2, a_1) = (-1)^{|a_1|} m(a_2, a_1) \end{array} \right\}$$

$$\circlearrowleft$$

$$\mathcal{G}(A) = \left\{ (G^d) \;\middle|\; G^1 = id,\ G^2 : V \otimes V \longrightarrow V[-1], \right.$$
$$\left. G^3 : V \otimes V \otimes V \longrightarrow V[-2], \ldots \right\}$$

Notation: Given $\mathcal{A} \in \mathcal{U}(A)$, $\mathcal{G} = (\mathcal{G}^d)_{d \geq 1} \in \mathcal{G}(A)$

$\mathcal{A}' = \mathcal{G}^* \mathcal{A}$:= unique A_∞ structure obtained by solving A_∞ functor equations recursively.

$$\mu'_k(\underset{G_1(x_1)}{x_1}, \dots, \underset{G_1(x_k)}{x_k}) =$$

$$\mu^d \rightsquigarrow (\varepsilon^* \mu)^d = \varepsilon^{d-2} \mu^d$$

$$\mathcal{U}(A) / \mathcal{G}(A) \underset{\text{biject}}{=} \frac{\{\text{pairs } (\mathcal{B}, F : A \xrightarrow{\sim} H(\mathcal{B}))\}}{\left\{ A_\infty \text{ quasi-iso } \mathcal{F} : \mathcal{B} \xrightarrow{\mathcal{F}} \mathcal{B}' \quad \begin{array}{ccc} H(\mathcal{B}) & \xrightarrow{H(\mathcal{F})} & H(\mathcal{B}') \\ F \downarrow & & \downarrow F \\ A & \xrightarrow[\text{id}]{} & A \end{array} \right\}}$$

homological perturbation

Obs. Hochschild cohomology of graded algebra (A, m) is bigraded

$$CC^{s+t}(A, A)^t = \left\{ \begin{array}{c} \text{multi-linear} \\ \tau : V^{\otimes s} \longrightarrow V \\ \text{of deg } t \end{array} \right\}$$

the differential $\partial^A_{Hoch} = [m, \cdot]$ preserves bigrading $\Rightarrow HH^{s+t}(A, A)^t$

Def. Let $\mathcal{A} \in \mathcal{U}(A)$ (denoted $\mu^s_{\mathcal{A}}$)

such that $\mu^s = 0$ for $2 < s < d$.

Then $\mu^d_{\mathcal{A}}$ is a cocycle for

Hochschild differential $\partial = [m, \cdot]$

Call its class

$$\mathcal{O}^d_{\mathcal{A}} = [\mu^d_{\mathcal{A}}] \in HH^2(A,A)^{2-d}$$

Thm. If $HH^2(A,A)^{2-s} = 0$

for all $2 < s$, then

$$\mathcal{U}(A)/_{\mathcal{G}(A)} = \text{base pt}$$

A is intrinsically formal:

Any A_∞-algebra $\mathcal{B}$ with

$G : H(\mathcal{B}) \to A$ as graded algebra

is actually quasi-iso to A

(with trivial A_∞ str.)

Thm. Suppose there exists some $d>2$ s.t.

$$\begin{cases} HH^2(A,A)^{2-d} \cong \mathbb{C} \\ HH^2(A,A)^{2-s} \cong 0, \quad s>2, \ s\neq d \end{cases}$$

Suppose $\exists\, \mathcal{A} \in \mathcal{U}(A)$ s.t. $\mu^s_{\mathcal{A}} = 0$ for $2<s<d$ and $o^d_{\mathcal{A}}$ is non-zero.

Then any $\mathcal{A}' \in \mathcal{U}(A)$ is equiv to $\varepsilon^* \mathcal{A}$ for some $\varepsilon \in \mathbb{C}$.

HKR iso:

Thm (HKR, 1962) Let A comm $\mathbb{K}$-alg, $X = \operatorname{Spec}(A)$ affine, smooth alg. var. of finite type. Then $\exists$ iso

$$HH(A) \cong H^\circ(X, \Lambda^\circ T_X)$$

$\mathbb{K} = \mathbb{C}$

Thm. Let V be a fixed v.s.

Take $A = \Lambda^\circ V = \mathbb{C}[\theta_1, \ldots, \theta_n]$

Let $\operatorname{Sym} V^\vee = \mathbb{C}[[z_1, \ldots, z_n]]$

Then $\phi_{HKR} : CC^*(A,A) \longrightarrow K[[z_1,\dots,z_n]][\theta_1,\dots,\theta_n]$

$$= \mathrm{Sym}^s V^\vee \otimes \Lambda^{s+t} V$$

$$\tau \longmapsto \sum_s \tau^s(z,\dots,z)$$

$$z = \sum z_i \theta_i$$

induces an isom

$$[\phi_{HKR}] : HH^*(A,A) \to \mathrm{Sym}^s V^\vee \otimes \Lambda^{s+t} V$$