Blaier I

Note Title 6/10/2015

AC1 Abstract A_∞-categories

Assume $\mathbb{K}$ coeff field, all categories $\mathbb{K}$-linear

$[n]$ shifts down, $\mathcal{A}$, $\mathcal{B}$, A_∞-cat, F

A_∞ functor, $\mathrm{Fuk}(M)$.

Goal: Learn enough homological algebra

to make $D^\pi \mathrm{Fuk}(M) = \Pi(\mathrm{Tw}(\mathcal{A}))$

Def. A pre-category $\mathcal{A}$ is:

(i) a set of objects $\mathrm{ob}(\mathcal{A})$.

(ii) $\forall X, Y \in \mathrm{ob}(\mathcal{A})$ a graded vector space $\hom^*(X,Y)$

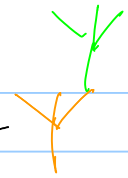
Def An A_∞-cat is a pre-category

(iii) $\forall d \geq 1$ there is

$(d=1)$ $\mu^1_{\mathcal{A}}: \hom(X_0,X_1) \to \hom(X_0,X_1)[1]$

$(d=2)$ $\mu^2_{\mathcal{A}}: \hom(X_1,X_2) \otimes \hom(X_0,X_1) \to \hom(X_0,X_2)$

$$\mu^d_{\mathcal{A}}: \hom_{\mathcal{A}}(X_{d-1},X_d) \otimes \dots \otimes \hom_{\mathcal{A}}(X_0,X_1) \to \hom_{\mathcal{A}}(X_0,X_d)[2-d]$$

$\forall\, a_d, \dots, a_1$ have $\quad \sum \pm$  $= 0$

$$\sum_{\substack{1\le l\le d \\ 0\le k\le d-l}} (-1)^{*} \mu^{d-k+1}\Big(a_d, \dots, a_{k+l+1},\ \mu^k_A\big(a_{k+l}, \dots, a_{k+1}\big), a_k, \dots, a_1\Big)$$

$$* = \sum_{i=1}^{k} \|a_i\| \qquad \|a_i\| = |a_i| - k.$$

What do the eqn tell us?

differential $\quad \mu^1(\mu^1(x)) = 0$

Leibnitz $\quad \mu^1(\mu^2(a,b)) = \pm\mu^2(\mu^1(b),a) \pm \mu^2(b,\mu^1(a))$

μ^2 is associative but only up to $\mu^1\mu^3$:

$$\mu^2(c,\mu^2(b,a)) \pm \mu^2(\mu^2(c,b),a) =$$
$$\pm\mu^1(\mu^3(c,b,a)) \pm \mu^2(\mu^1(c),b,a) \pm$$
$$\pm\mu^2(c,\mu^1(b),a) \pm \mu^2(c,b,\mu^1(a))$$

Bar complex

$$T(A[1]) = \bigoplus_{d\ge 0} A[1]^{\otimes d}$$

$$= K \oplus \left(\bigoplus_{d \geq 1} \bigoplus_{\substack{X_0, \dots, X_d \\ \in ob(A)}} hom^*(X_{d-1}, X_d) \otimes \cdots \otimes hom(X_0, X_1)[d] \right)$$

$$D_\mu = 1 + \Upsilon + \Psi + \dots$$

$$= \sum a_0 \otimes \cdots \otimes \mu^\ell(\text{—}) \otimes a_{k+m} \otimes \text{—} \otimes a_1$$

Example 1 Every dg-cat A is A_∞-cat

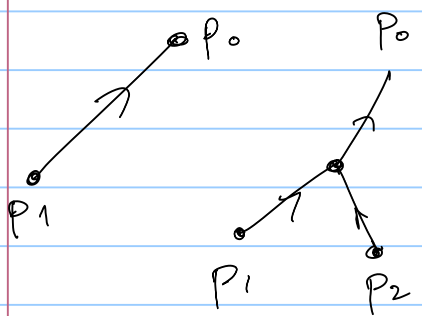
$$\partial a = (-1)^{|a|} \mu^1(a)$$

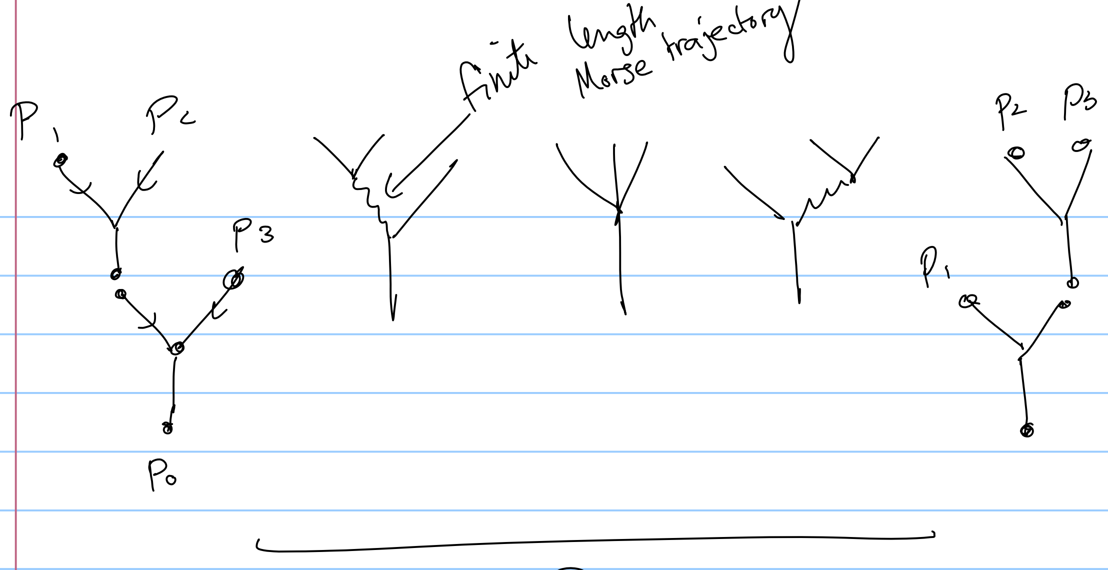
$$m(a_2, a_1) = (-1)^{|a_1|} \mu^2(a_2, a_1)$$

$$\mu^{\geq 3} = 0$$

Example 2 (M, f)

$CM^*(M, f) = Crit(f)$ graded by the Morse index.





$\mathbb{R}$

$\mathcal{M}_3$

$Fuk(M, \omega)$

(i) $ob(\mathcal{F}) =$ lag

(ii) $hom^*_{Fuk(M)}(L_0, L_1) = CF^*(L_0, L_1)$

$$\mu^s : hom^*(L_{s-1}, L_s) \otimes \cdots \otimes hom^*(L_0, L_1) \to_{M} hom^*(L_0, L_s)$$

$$\mu^s(p_1, \dots, p_s) = \sum \#\left(\quad \right) T^{\omega(f)} p_0$$

<u>Def</u>: Given $\mathcal{A}$ A_∞-cat, we can define the cohomological category $H(\mathcal{A})$

$$Ob_{H(\mathcal{A})} = ob_{\mathcal{A}}$$

$$Mor_{H(\mathcal{A})} = H^*(hom_{\mathcal{A}}(X_0, X_1), \mu^1_{\mathcal{A}})$$

composition

$$[a_2] \circ [a_1] = (-1)^{|a_1|} [\mu^2_{\mathcal{A}}(a_2, a_1)]$$

__Def__ If $H(\mathcal{A})$ has units, we say that $\mathcal{A}$ is c-unital.

__Assumption__: all A_∞-cat are c-unital

II) Functors

__Def.__ Let $\mathcal{A}, \mathcal{B}$ A_∞-cat.

A Functor $F : \mathcal{A} \to \mathcal{B}$

is (i) a map on objects

$$F : ob(\mathcal{A}) \to ob(\mathcal{B})$$

(ii) multi-linear maps

$$F^d : hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes hom_{\mathcal{A}}(X_0, X_1) \to$$
$$\to hom_{\mathcal{A}}(F(X_0), F(X_d))[1-d]$$

s.t.

$$\sum_r \sum_{s_1+\dots+s_r=d} \mu_B^r\left(F^{s_r}(a_d,\dots,a_{d-s_r+1}),\dots \right.$$
$$\left. \dots F^{s_1}(a_s,\dots,a_1)\right)$$

$$= \sum (-1)^{*} F^{d-l+1}\left(a_d,\dots,a_{k+l+1},\mu_A^l(a_{k+l},\dots,a_{k+1}),a_k,\dots,a_1\right)$$

<u>Bar complex interpretation</u>:

$$A \xrightarrow{F} B$$

$$TA \xrightarrow{TF} TB$$

$$a_d\otimes\dots\otimes a_1 \mapsto \sum F^{s_r}(a_d\otimes\dots)\otimes\dots\otimes F^{s_1}(\dots\otimes a_1)$$

$$TF\, D_A = D_B\, TF$$

<u>Observation</u>:

$$\mu_B^2(F^1 b, F^1 a) + \mu_B^1(F^2(b,a)) =$$
$$= F^1(\mu^2(b,a)) \pm F^2(b,\mu^1(a)) \pm$$
$$\pm F^2(\mu^1(b),a)$$

$\Longrightarrow F^1$ respects μ^2 up to homotopy by F^2.

<u>Def.</u> $F: \mathcal{A} \to \mathcal{B}$

$\rightsquigarrow \quad H(F): H(\mathcal{A}) \to H(\mathcal{B})$

obj : $X \longmapsto F(X)$

mor : $[a] \longmapsto [F^1(a)]$

is an ordinary non-unital functor.

<u>Def</u> F quasi-iso if $H(F)$ iso

F quasi-equiv if $H(F)$ equiv

<u>Rmk</u> composition of A_∞-functors

$$(G \circ F)^d = \sum_{\substack{r \\ s_1 + \dots + s_r = d}} G^r(F^{s_r}(-), \dots, F^{s_1}(-))$$

is strictly associative

Given two A_∞ cat $\mathcal{A}, \mathcal{B}$

$Fun(\mathcal{A}, \mathcal{B})$:

obj: functors

mor: pre-natural transformations

$T \in hom^g(\mathcal{F}, \mathcal{G})$:

$T^0 = \{ \forall X \in ob(\mathcal{A}) \mapsto$ a map $\in hom^g_{\mathcal{B}}(\mathcal{F}(X), \mathcal{G}(X)) \}$

but also

T^d: $hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes hom_{\mathcal{A}}(X_0, X_1) \to$

$\to hom_{\mathcal{B}}(\mathcal{F}(X_0), \mathcal{G}(X_d))[1-d]$

Can define operation $\rightsquigarrow$ A_∞-category

$\mu^1, \mu^2, \ldots$

on natural

transformations

$\Rightarrow A_\infty$ cat

III) exact triangles

Want to define derived category

Abelian

$$\mathcal{A} \to C(\mathcal{A}) \to K(\mathcal{A})$$
$$\to D(\mathcal{A})$$

A_∞ world

$$\mathcal{A} \to Tw\,\mathcal{A}$$
$$\mathcal{A} \to Tw\,\mathcal{A} \to \Pi(Tw\,\mathcal{A}) \to D^{\Pi}(\mathcal{A})$$

T_3 : - category

obj: $\{Z_0, Z_1, Z_2\}$

mor: dim 1 v.s.

$$\begin{tikzcd}
Z_0 \arrow[loop left, "ke_{Z_0}"] \arrow[rr, "k x_1"', "|x_1|=0"] & & Z_1 \arrow[loop right, "ke_{Z_1}"] \arrow[dl, "k x_2 \quad |x_2|=0"] \\
& Z_2 \arrow[loop below, "ke_{Z_2}"] \arrow[ul, "k x_3 \quad |x_3|=1"] &
\end{tikzcd}$$

Def. An exact triangle in $H^0(\mathcal{A})$ is a diagram

$$\begin{tikzcd} Y_0 \arrow[rr, "{[c_1]}"] & & Y_1 \arrow[dl, "{[c_2]}"] \\ & Y_2 \arrow[ul, "{[c_3]}"] & \end{tikzcd}$$

such that $\exists$ A_∞-functor

$$F : T_3 \to \mathcal{A}$$

with

$$H(F)(z_k) = Y_k$$

$$H(F)(x_k) = [c_k].$$

Cor. If $F : \mathcal{A} \to \mathcal{B}$ A_∞-functor then $H(F)$ sends exact triangles to exact triangles

Def. An A_∞ cat is <u>triangulated</u> if it is "closed under shifts" and every morphism $[c_1] \in H^0(\mathcal{A})$ can be completed to an exact triangle

Thm. If $\mathcal{A}$ is triangulated, $H^0(\mathcal{A})$ satisfies (TR1) --- (TR4) (usual axioms of triangulated category)

<u>Twisted complexes</u> two stages

1) additive enlargement $\Sigma\mathcal{A}$

allow formal direct sums

<u>Ob:</u> $\bigoplus_{\text{finite}} X_i[K_i]$ $\quad X_i \in ob(\mathcal{A})$, $K_i \in \mathbb{Z}$.

<u>Mor:</u> $hom\left(\bigoplus_{i=1}^{n} X_i[k_i], \bigoplus_{j=1}^{m} Y_j[l_j]\right) =$

$$= \bigoplus_{i,j} hom_{\mathcal{A}}(X_i, Y_j)[l_j - k_i]$$

$$= \left\{ a = \begin{pmatrix} & \vdots & \\ \cdots & a^{ij} & \cdots \\ & \vdots & \end{pmatrix} \right\}$$

multiplication is

$$\mu^k_{\Sigma A}\left(\left(\quad\right), \text{---} \left(\quad\right)\right)_{i_0 i_k}$$

$$= \sum_{i_1,\dots,i_{k-1}} \mu^k_A\left(a_k^{i_{k-1} i_k}, \dots, a_1^{i_0 i_1}\right)$$

2) Tw A

ob (X, δ_X) where

$X = \bigoplus_{\text{finite}} X_i[K_i] \in ob_{\Sigma A}$

$\delta_X = \delta_X^{ij} \in hom_{\Sigma A}(X, X)$ s.t.

$$\begin{cases} \delta_X \text{ is strictly lower triangular} \\ \sum_{k=1}^{\infty} \mu^k(\delta_X, \dots, \delta_X) = 0 \end{cases}$$

Mor

$hom_{Tw(A)}((X,\delta_X),(Y,\delta_Y)) =$

$= hom_{\Sigma A}(X, Y)$

multiplication

$$\mu^k_{Tw(A)}(a_k, \dots, a_1) =$$

$$\sum_{i_0,\dots,i_k} \mu^{k+i_0+\dots+i_k}_{\Sigma A}\Big(\underbrace{\delta_{X_k},\dots,\delta_{X_k}}_{i_k}, a_k, \underbrace{\delta_{X_{k-1}},\dots,\delta_{X_{k-1}}}_{i_{k-1}},$$

$$\cdots\; a_1, \underbrace{\delta_{X_0},\dots,\delta_{X_0}}_{i_0}\Big)$$