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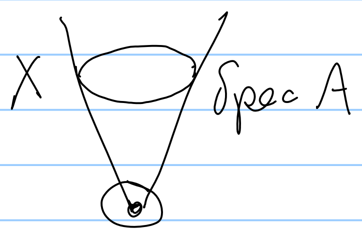
Note Title 12/06/2015

$$\begin{tikzcd}
 & D^b(\text{gr-} S/W) \arrow[dl, shift right] \arrow[dr, shift left] & \\
\mathrm{Gr}\,MF(S,W) \simeq D^{gr}_{sg}(\underbrace{S/W}_{A}) \arrow[ur, shift right] & & D^b(\mathrm{Coh}\, X) \arrow[ul, shift left]
\end{tikzcd}$$

$$\underset{\substack{\shortparallel \\ \text{smooth} \\ \text{projective} \\ \text{variety}}}{X} = \mathrm{Proj}\ A = (\mathrm{Spec}\ A \setminus 0) / \underbrace{G_m}_{\substack{\text{multiplicative group of field} \\ \shortparallel \\ \mathbb{C}^\times}}$$



$$\text{Sheaves on } X \longleftrightarrow \underbrace{\mathbb{C}^\times \text{ equivariant sheaves on Spec } A}_{\text{graded } A\text{-modules}} \Big/ \underbrace{\substack{\text{sheaves} \\ \text{supported} \\ \text{at } 0}}_{\text{torsion modules}}$$

$$\begin{tikzcd}
D^b(\text{gr-}A)/\text{tors} \arrow[d, "\simeq"] \\
D^b(\mathrm{coh}\, X)
\end{tikzcd}$$

$$\begin{tikzcd} & D^b(\text{gr-}A_{\geq 0}) \arrow[dl, shift left] \arrow[dr, shift left] & \\ D^{gr}_{sg}(A) \arrow[ur, shift left, "\text{left adjoint}"'] & & D^b(\text{Coh}\, X) \arrow[ul, shift left, "\text{right adjoint}"'] \end{tikzcd}$$

Semi-orthogonal Decomposition

$\mathcal{D}$ triangulated category

$\mathcal{A} \subset \mathcal{D}$ triangulated sub-category

$$\mathcal{D} \xrightarrow{p} \mathcal{D}/\mathcal{A}$$

Define ${}^\perp\mathcal{A} = \{ B \in \mathcal{D} \mid \mathrm{Hom}(B, A) = 0 \ \forall A \in \mathcal{A} \}$

Lemma: p has a right adjoint

$$\Updownarrow$$

$\forall X \in D$: $\exists$ exact triangle

$$\begin{tikzcd} B \arrow[r] & X \arrow[r] & A \\ {}^\perp\mathcal{A} \arrow[u, phantom, "\cap" rotate=90] & & \mathcal{A} \arrow[u, phantom, "\cap" rotate=90] \end{tikzcd}$$

Right adjoint: $\mathcal{D}/\mathcal{A} \longrightarrow \mathcal{A}$

$$[X] \rightsquigarrow B$$

In this situation, we say

$$\mathcal{D} = \langle {}^{\perp}\mathcal{A}, \mathcal{A} \rangle$$

and we have

$${}^{\perp}\mathcal{A} \xrightarrow{\sim} \mathcal{D}/\mathcal{A}$$

In our case

torsion modules

$$D(\mathrm{gr}\text{-}A_{\geq 0}) = \langle D_0, S_{\geq 0} \rangle$$

$$D_0 \cong D^b(\mathrm{Coh}\,X)$$

related by duality

perfect complexes

$$D(\mathrm{gr}\text{-}A_{\geq 0}) = \langle P_{\geq 0}, T_0 \rangle$$

$$T_0 \cong D^{gr}_{sg}(A)$$

since X is Calabi-Yau

Theorem

$$A = k[x_1, \dots, x_n] / \textstyle\sum x_i^n$$

Idea of proof:

$$\begin{tikzcd} & D^b(\mathrm{gr}\text{-}A) \arrow[dl, "\text{has r. adjoint}"'] \arrow[dr, "\text{has l. adj.}"] & \\ D^{gr}_{sg}(A) & & D^b(\mathrm{Coh}\,X) \end{tikzcd}$$

$P_{\geq 0}$ – perfect complexes in $\deg \geq 0$
generated by $A(\underset{\sim}{r})$ $r \leq 0$
shift grading by r.

$\mathrm{Hom}(A(r), M) = M_{-r}$

$S_{\geq 0}$ – torsion complexes
gen. by $k(r)$, $r \leq 0$.

$\mathrm{Hom}(M, k(r)) = M^*_{-r}$

$R\mathrm{Hom}(k, A) = k[-n]$

$\iff \mathrm{Ext}^m(k, A) = 0$
unless $m = n$:
$\mathrm{Ext}^n(k, A) = k$

Compute image of $k \in D^{gr}_{sg}(A)$
in $D^b(\mathrm{Coh}\, X)$:
we need an exact triangle

$$\begin{array}{ccccc} t & \longrightarrow & k & \longrightarrow & P \\ \cap & & & & \cap \\ T_0 & & & & P_{\geq 0} \end{array} \qquad (*)$$

Then in $D^b(\mathrm{Coh}\, X)$, k maps to $P[\pm 1]$.
Indeed, projecting triangle $(*)$ sends k to 0
So t, which corresponds to k under projection to D^{gr}_{sg},
is equivalent to $P[\pm 1]$.

t has to satisfy

$$\mathrm{Ext}^m(t, A(r)) = 0 \quad \forall m, \ \forall r \leq 0$$

$$\Longrightarrow \mathrm{Ext}^m(p, A(r)) = \mathrm{Ext}^m(K, A(r)) \quad \forall m, \ \forall r \leq 0$$

but $\mathrm{Ext}^m(K, A(r)) = 0$

unless $r = 0$, $m = n$

<u>Guess</u>: Set $P = A[-n]$

$$K \longrightarrow A[-n]$$

from $\mathrm{Ext}^n(K, A)$

$\Longrightarrow$ image of $k = [A[-n]]$

in geometric terms $\mathcal{O}_X[-n]$

Compute images of $K(r)$ $0 < r < n$

Consider a free resolution of k as an A-module

$$C \xrightarrow{\sim} k$$

$\rightsquigarrow \quad C(r) \xrightarrow{\sim} k(r)$

<u>Claim:</u> $D(\text{gr-}A) = \langle D(\text{gr-}A_{\geq 0}), P_{<0} \rangle$

$$\begin{array}{c} C(r)_{<0} \longrightarrow C(r) \longrightarrow C(r)_{\geq 0} \\ \qquad\qquad\; \wr\! \text{S} \\ \qquad\qquad\; K(r) \end{array}$$

<u>Claim:</u> $C(r)_{\geq 0} \in T_0$

$\Rightarrow$ image of $k(r)$

$\simeq C(r)_{<0}[\pm 1]$

$$A = k[x_1, \dots, x_n]/W$$

$C \xrightarrow{\sim} k$ as an A-mod

unbounded because A singular

can be obtained from Koszul resolution K by quotient $/(W)$ and resolving resulting kernel at last term

$K \longrightarrow k$ - as a $k[x_1, \dots, x_n]$ module

Thus,

$C(r)_{<0} \simeq K(r)_{\text{cut off at } r.}$