Three Envelopes*

Sergiu Hart†

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Consider the variant of the so-called "secretary problem" where the realizations of the random variables—and not only their relative rankings—are sequentially observed. This is sometimes called "googol"; see Ferguson (1989) and Gnedin (1994) (whose general solution implies the result below). We will provide here an elementary proof for $n = 3$.

Let $x < y < z$ be the 3 numbers. Let W_1, W_2, W_3 be the triple x, y, z in a *random order* (thus $\mathbf{P}(W_1 = x, W_2 = y, W_3 = z) = \cdots = \mathbf{P}(W_1 = z, W_2 = y, W_3 = x) = 1/6$)).

A strategy σ consists of 2 functions:

$$\begin{aligned} \alpha &: \mathbb{R} \to [0,1]; \\ \beta &: \mathbb{R} \times \mathbb{R} \to [0,1], \end{aligned}$$

where

$$\begin{aligned} \alpha(w_1) &:= \mathbf{P}_\sigma(\text{KEEP } W_1 \,|\, W_1 = w_1), \text{ and} \\ \beta(w_1, w_2) &:= \mathbf{P}_\sigma(\text{KEEP } W_2 \,|\, W_1 = w_1 \text{ was not kept}, W_2 = w_2). \end{aligned}$$

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†Department of Mathematics, Department of Economics, and Center for the Study of Rationality, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel. *e-mail:* `hart@huji.ac.il` *web page:* `http://www.ma.huji.ac.il/hart`

Let W^* be the W_i that is eventually kept. A "WIN" is defined as the event that $W^* = \max\{W_1, W_2, W_3\}$.

Proposition 1 *Assume that* $\sigma = (\alpha, \beta)$ *satisfies* $\mathbf{P}_\sigma(\mathit{WIN}) \geq 1/2$ *for every* $x < y < z$. *Then:*

(i) $\alpha(w) = 0$ *for every* w;

(ii) $\beta(w_1, w_2) = 1$ *for all* $w_2 > w_1$, *and* $\beta(w_1, w_2) = 0$ *for all* $w_2 < w_1$;

(iii) $\mathbf{P}_\sigma(\mathit{WIN}) = 1/2$ *for every* $x < y < z$.

This shows that the best one can obtain uniformly is $\mathbf{P}_\sigma(\text{WIN}) = 1/2$. We now provide an elementary proof.

Lemma 2 *Without loss of generality* $\beta(w_1, w_2) = 0$ *for all* $w_2 < w_1$.

Proof. Decreasing β to 0 when $w_2 < w_1$ can only increase the probability of WIN. □

Lemma 3 *For every* $x < y < z$:

$$\begin{aligned} 3 \;\leq\; & (1 - \alpha(x))(1 - \beta(x, y) + \beta(x, z)) \\ & + (1 - \alpha(y))(1 + \beta(y, z)) + 2\alpha(z). \end{aligned} \tag{1}$$

Proof. The right-hand side is $6\mathbf{P}(\text{WIN})$ (add the probability of WIN for each one of the 6 orders); now use the assumption that $\mathbf{P}_\sigma(\text{WIN}) \geq 1/2$. □

Lemma 4 $\alpha(w) = 0$ *for every* w.

Proof. Fix x. Let $(y_n)_{n=1,2,\ldots}$ be a *strictly decreasing* sequence (i.e., $y_{n+1} < y_n$ for all n), with limit $y > x$. By taking a subsequence, assume that $\alpha(y_n) \to r$ and $\beta(x, y_n) \to s$ for some r, s. Consider the triple $x < y_{n+1} < y_n$, then Lemma 3 and $\beta(y, z) \leq 1$ imply

$$\begin{aligned} 3 \;&\leq\; (1 - \alpha(x))(1 - \beta(x, y_{n+1}) + \beta(x, y_n)) + (1 - \alpha(y_{n+1}))(1 + 1) + 2\alpha(y_n) \\ &\to\; (1 - \alpha(x))(1 - s + s) + 2(1 - r) + 2r = 3 - \alpha(x). \end{aligned}$$

Therefore $\alpha(x) \le 0$. □

Lemma 5 $\beta(w_1, w_2) = 1$ *for all* $w_2 > w_1$.

Proof. Lemma 4 and (1) imply $3 \le 1 - \beta(x,y) + \beta(x,z) + 1 + \beta(y,z)$, or

$$\beta(x,z) - \beta(x,y) + \beta(y,z) \ge 1, \tag{2}$$

for every $x < y < z$. Since $\beta(y,z) \le 1$, it follows that $\beta(x,z) - \beta(x,y) \ge 0$, and so $\beta(x,\cdot)$ is a monotonically nondecreasing function for every x.

Fix $x < y$. Let $z \to y^+$ (i.e., z decreases to y); from (2) we get

$$\beta(x,y^+) - \beta(x,y) + \beta(y,y^+) \ge 1, \tag{3}$$

where $\beta(x,y^+) := \lim_{z\to y^+} \beta(x,z)$ (recall that $\beta(x,\cdot)$ is monotonic). Now $\beta(x,y)$ is bounded (in $[0,1]$), so $\beta(x,y^+) - \beta(x,y) = 0$ for all except at most countably many $y > x$. Let $A \equiv A_x$ be the set of all those y; then (3) implies $\beta(y,y^+) = 1$ for all $y \in A$, hence $\beta(y,z) = 1$ for all $z > y \in A$ (by monotonicity).

Let y and z be such that $x < y < z$. Then there exists $y' \in A$ with $x < y' < y$, and (2) for $y' < y < z$ yields $1 - 1 + \beta(y,z) \ge 1$, or $\beta(y,z) = 1$. Now x was arbitrary. □

References

[1] Ferguson, T. S. (1989), "Who Solved the Secretary Problem," *Statistical Science* 4, 3, 282–296.

[2] Gnedin, A. V. (1994), "A Solution to the Game of Googol," *The Annals of Probability* 22, 3, 1588–1595.