Arthur-Selberg trace formula

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Selberg's trace formula

Notation

$\mathbb{H}$ – the hyperbolic plane $\mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}(2)$

$\Delta = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ – the hyperbolic Laplacian

Γ – a lattice in $\mathrm{SL}_2(\mathbb{R})$

Δ descends to $\Gamma\backslash\mathbb{H}$

$0 = \lambda_0 < \lambda_1 \leq \dots$ be eigenvalues of Δ on $L^2(\Gamma\backslash\mathbb{H})$

Write $\lambda_j = \frac{1}{4} + r_j^2$, $j = 0, 1, \dots$

c – number of cusps

$\Phi(s)$ – scattering matrix (of size $c \times c$)

$\phi(s) = \det \Phi(s)$

$h \in C_c^\infty(\mathbb{R})^{\text{even}}$

$\hat{h}(r) = \int_{-\infty}^{\infty} e^{irt} h(t)\, dt$

For any $h \in C_c^\infty(\mathbb{R})^{\text{even}}$

$$\sum_j \hat{h}(r_j) - \frac{1}{4\pi}\int_{-\infty}^{\infty} \hat{h}(r)\frac{\phi'}{\phi}(\frac{1}{2}+\mathrm{i}r)\ dr + \frac{\hat{h}(0)}{4}\,\mathrm{Tr}\,\Phi(\frac{1}{2}) =$$

$$\frac{\mathrm{Area}(\Gamma\backslash\mathbb{H})}{4\pi}\int_{-\infty}^{\infty} \hat{h}(r)r\tanh(\pi r)\ dr +$$

$$\sum_P \sum_{\ell=1}^{\infty} \frac{\log p}{p^{\ell/2}-p^{-\ell/2}} h(\ell \log p) +$$

$$\sum_{\mathcal{R}} \sum_{0<\ell<m} (2m\sin\frac{\pi\ell}{m})^{-1}\int_{-\infty}^{\infty} \hat{h}(r)\frac{\cosh\pi(1-2\ell/m)r}{\cosh\pi r}\ dr +$$

$$\frac{c}{4}\hat{h}(0) - ch(0)\log 2 - \frac{c}{2\pi}\int_{-\infty}^{\infty} \hat{h}(r)\frac{\mathbf{\Gamma}'}{\mathbf{\Gamma}}(1+\mathrm{i}r)\ dr$$

where P ranges over primitive hyperbolic classes of norm $p = NP$
$\mathcal{R}$ ranges over primitive elliptic classes of order m

Special case

If Γ is torsion free (no elliptic elements) and co-compact ($c = 0$) we remain with

$$\sum_j \hat{h}(r_j) = \frac{\text{Area}(\Gamma\backslash\mathbb{H})}{4\pi} \int_{-\infty}^{\infty} \hat{h}(r) r \tanh(\pi r)\; dr + \sum_P \sum_{\ell=1}^{\infty} \frac{2\log p}{p^{\ell/2} - p^{-\ell/2}} h(\ell \log p)$$

Weyl's law

Weyl's original result.

Let Ω be a bounded region in the Euclidean plane with smooth boundary $\partial\Omega$. Consider the Euclidean Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ on the plane. We want to count solutions to the differential equation

$$\Delta\phi + \lambda\phi = 0, \quad \lambda \geq 0$$

with Dirichlet boundary condition $\phi|_{\partial\Omega} = 0$. Let $N_\Omega(R)$ be the counting function for the number of linearly independent solutions with $\lambda \leq R$. Weyl's result is

$$N_\Omega(R) \sim \frac{\text{Area}(\Omega)}{4\pi} R \text{ as } R \to \infty.$$

In a more modern setting we consider a compact Riemannian d-manifold (M, g) and its Laplacian $\Delta = \operatorname{div}\operatorname{grad}$. This is a negative definite (unbounded) operator on $L^2(M, g)$. Let N_T be the counting function for the number of eigenfunctions with $\lambda \leq T^2$. In this context Weyl's law is

$$N_T \sim \frac{\operatorname{vol}(M)}{(4\pi)^{d/2}\Gamma(d/2+1)} T^d \text{ as } T \to \infty.$$

(In a more general context – *Hörmander*)

Exmaples

Example 1: the sphere S^2. The eigenfunctions of Δ are the spherical harmonics. The eigenvalues are $n(n+1)$, $n = 0, 1, 2, \ldots$ with multiplicity $2n+1$.

Example 2: the two-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$. Once again, we can write down explicitly the eigenfunctions, in this case as periodic exponential functions. The eigenvalues are $n^2 + m^2$. We can approximate

$$N_R = \#\{(n, m) : n^2 + m^2 < R^2\}$$

by the area of the circle of radius R. A trivial upper bound for the remainder term is $O(R)$. To find the exact order of magnitude of the remainder is a much more serious question known as the Gauss circle problem.

Compact hyperbolic surfaces

$M = \Gamma\backslash\mathbb{H}$, Γ is a uniform lattice of $SL(2,\mathbb{R})$. Let $\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues of Δ acting in $L^2(\Gamma\backslash\mathbb{H})$. It is useful to set $\lambda_j = \frac{1}{4} + r_j^2$ with $r_j \in \mathbb{R}_{\ge 0} \cup [0, \frac{1}{2}]\mathrm{i}$, and $r_{-j} = -r_j$. Thus, $N_T = \#\{j : |r_j| \le T\}$. For $h \in C_c^\infty(\mathbb{R})^{\text{even}}$ supported near 0 the Selberg trace formula simplifies to

$$\sum_{j=-\infty}^{\infty} \hat{h}(r_j) = \frac{\text{Area}(\Gamma\backslash\mathbb{H})}{2\pi} \int_{\mathbb{R}} \hat{h}(r) r \tanh(\pi r)\, dr.$$

Take $h \ge 0$ such that $h(0) = 1$, $\hat{h} \ge 0$ on $\mathbb{R} \cup [-\frac{1}{2}, \frac{1}{2}]\mathrm{i}$ and $\hat{h} \ge 1$ on $[-1, 1]$. For $t \in \mathbb{R}$ let

$$h_t(a) = \frac{1}{2} h(a)(e^{\mathrm{i}at} + e^{-\mathrm{i}at})$$

(still supported near 0!) so that

$$\widehat{h_t}(r) = \frac{1}{2}(\hat{h}(t-r) + \hat{h}(t+r)).$$

Applying the trace formula to h_t we get

$$\sum_j \hat{h}(t-r_j) = \frac{\text{Area}(M)}{4\pi}\int_{\mathbb{R}} \hat{h}(t-r) r \tanh(\pi r)\, dr. \qquad (1)$$

Since $|\tanh| \leq 1$ the right-side is $O(T)$. We infer the local estimate

$$\#\{j : |r_j - T| \leq 1\} = O(T).$$

Let μ be a measure on $\mathbb{R}$ such that $\mu([T, T+1]) = O(T)$. Then

$$\int_{-T}^{T}\int_{\mathbb{R}} \hat{h}(t-r)\, d\mu(r)\, dt = \int_{\mathbb{R}}\int_{-T}^{T} \hat{h}(t-r)\, d\mu(r)\, dt -$$

$$\int_{|t|>T}\int_{-T}^{T} \hat{h}(t-r)\, d\mu(r)\, dt + \int_{-T}^{T}\int_{|r|>T} \hat{h}(t-r)\, d\mu(r)\, dt$$

$$= \mu([-T,T])\int_{\mathbb{R}} \hat{h}(t)\, dt + O(T) = \mu([-T,T]) + O(T). \quad (2)$$

Integrating (1) over $t \in [-T, T]$ and using (2) for $\mu = \sum_j \delta_{r_j}$ and $\mu = r\tanh(\pi r)\, dr$ we get

$$LHS = N_T + O(T), \quad RHS = \frac{\text{Area}(M)}{4\pi} T^2 + O(T), \qquad \text{as required.}$$

If Γ is not co-compact, the same argument works, but we have to take into account the additional terms in the trace formula. By Stirling's formula the term

$$\int_{-\infty}^{\infty} \hat{h}(r) \frac{\Gamma'}{\Gamma}(1 + \mathrm{i}r) \, dr$$

gives $2T \log T$. The other terms on the geometric side are negligible. We get

$$N_T + M_T = \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{4\pi} T^2 - \frac{c}{\pi} T \log T + O(T)$$

where

$$M_T = -\frac{1}{4\pi} \int_{-T}^{T} \frac{\phi'}{\phi}(\frac{1}{2} + \mathrm{i}t) \, dt$$

is the winding number of ϕ. Roughly, M_T counts the number of poles of ϕ (on $\text{Re}(s) < 0$) with imaginary part $< T$. This is the contribution of the continuous spectrum.

Question: Is M_T negligible with respect to N_T or the other way around?

The case of $\Gamma = SL(2,\mathbb{Z})$

This is a case of a non-uniform lattice. The fundamental domain is the familiar hyperbolic triangle

$$\{z \in \mathbb{H} : |z| \geq 1 \text{ and } |\mathrm{Re}\, z| \leq 1/2\}.$$

In this case there is one cusp ($c = 1$).

Eisenstein series

$$E(z;s) = \sum_{(m,n)=1} \frac{y^{s+\frac{1}{2}}}{|mz+n|^{2s+1}} = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y(\gamma z)^{s+\frac{1}{2}} \quad z \in \mathbb{H}$$

where $\Gamma_\infty = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$. The series converges for $\mathrm{Re}\, s > \frac{1}{2}$ and $E(\gamma z; s) = E(z;s)$ for all $\gamma \in \Gamma$. Also, $\Delta E(\cdot; s) = (\frac{1}{4} - s^2) E(\cdot; s)$ since this is true for the function $y^{s+\frac{1}{2}}$ and E is obtained from it by averaging. Less evident is the fact that $E(z, \cdot)$ admits meromorphic continuation to $\mathbb{C}$ and a functional equation

$$E(z;s) = \phi(s) E(z;-s), \quad \phi(s) = \frac{\sqrt{\pi}\,\boldsymbol{\Gamma}(s)\zeta(2s)}{\boldsymbol{\Gamma}(s+\frac{1}{2})\zeta(2s+1)}.$$

So in this case

$$M_T = O(T \log T).$$

More generally, the same holds if Γ is a congruence subgroup (ϕ is order one) and therefore the Weyl law holds.

Recall that we do not know how to write down explicitly a single Maass form for $SL_2(\mathbb{Z})$ in a closed form.

What happens for non-uniform lattices which are not congruence subgroups?

Selberg believed that the Weyl law should hold for them as well. However, the work of *Phillips* and *Sarnak* on the dissolution of cusp forms under deformation of congruence subgroups suggests quite the contrary. In fact, it may very well be the case that the discrete spectrum is finite for a generic Γ in a family! However, although there are cogent theoretical (and empirical?) results (Luo, Wolpert,..) there are no known unconditional results.

Prime Geodesic Theorem

Take $h(t) = (e^{t/2} + e^{-t/2})\psi(t)$ where

$$\psi(t) = \begin{cases} 1 & 0 \le t \le \log X \\ f(\frac{t-\log X}{\log(X+Y)-\log X}) & \log X \le t \le \log(X+Y) \\ 0 & t > \log(X+Y) \end{cases}$$

where $f \in C_c^\infty(\mathbb{R})$, $f(0) = 1$, $f(x) = 0$ for $x \ge 1$ and $Y \le X$.
Then $\psi^{(n)}(t) = O(T^n)$, $T = X/Y$ (supported in $[\log X, \log(X+Y)]$, an interval of size T^{-1}).
$|\hat{h}(r)| \ll \sqrt{X}/|r| \min(1, (T/|r|)^2)$, $r \in \mathbb{R}$
$\hat{h}(\mathrm{i}r) = \frac{X^{\frac{1}{2}+r}}{\frac{1}{2}+r} + O(Y + \sqrt{X})$, $r \in [0, \frac{1}{2}]$
Thus,

$$\sum_j \hat{h}(r_j) = \sum_{j:\frac{1}{2}<s_j\le 1} X^{s_j}/s_j + O(Y + \sqrt{X}T)$$

(where $s_j = \frac{1}{2} - \mathrm{i}r_j$)
$\int_{-\infty}^{\infty} \hat{h}(r) r \tanh(\pi r)\, dr, \int_{-\infty}^{\infty} \hat{h}(r) \frac{\phi'}{\phi}(\frac{1}{2} + \mathrm{i}r)\, dr \ll \sqrt{X}T$

We get

$$\sum_{P} \psi(\log p) \log p = \sum_{\frac{1}{2}<s_j\leq 1} X^{s_j}/s_j + O(Y + \sqrt{X}T).$$

This gives

$$\sum_{X<p<X+Y} \log p = O(Y + \sqrt{X}T).$$

Taking $Y = X^{3/4}$ we finally get

$$\sum_{p\leq X} \log p = X + \sum_{\frac{1}{2}<s_j<1} X^{s_j}/s_j + O(X^{3/4}).$$

Note For congruence subgroups Selberg showed that $s_j \leq 3/4$.
It is now known that $s_j \leq \frac{1}{2} + 7/64$. (*Kim-Sarnak*)
It is also known that

$$\sum_{p\leq X} \log p = X + O(X^{7/10+\epsilon}) \quad \forall \epsilon > 0$$

(*Luo-Sarnak*)

Selberg's trace formula – Group theoretic formulation

G – any Lie group, dg Haar measure, Γ a lattice in G. That is, Γ is discrete and $X = \Gamma\backslash G$ has finite volume. In particular, G is unimodular. Let R be the regular representation of G on $L^2(X)$

$$[R(g)\phi](x) = \phi(xg) \quad g \in G, x \in X.$$

We can extend R to bounded measures on G and in particular to a representation of the Banach algebra $L^1(G)$ (with respect to convolution) by

$$R(f)\phi(x) = \int_G f(g)\phi(xg)\, dg = \int_G f(x^{-1}g)\phi(g)\, dg.$$

Suppose that $f \in C_c^\infty(G)$. By splitting the integral, we can write

$$R(f)\phi(x) = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma g)\phi(g)\, dg = \int_{\Gamma\backslash G} K_f(x,y)\phi(y)\, dy.$$

Thus, $R(f)$ is an integral operator with smooth kernel

$$K_f(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y) \quad x, y \in X. \tag{3}$$

Suppose that X is compact, i.e., Γ is uniform

Then $R(f)$ is trace class. We can compute its trace in two ways. First we write

$$\operatorname{tr} R(f) = \int_{\Gamma\backslash G} K_f(x,x)\, dx = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)\, dx$$

and break the sum over γ into conjugacy classes of Γ. The conjugacy class $[\gamma] = \{\delta^{-1}\gamma\delta : \delta \in \Gamma_\gamma\backslash\Gamma\}$ where Γ_γ is the centralizer of γ in Γ contributes

$$\int_{\Gamma\backslash G} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\, dx = \int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)\, dx = \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) I(\gamma, f)$$

where $I(\gamma, f)$ is the orbital integral

$$I(\gamma, f) = \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\, dx.$$

Note that Γ_γ is a uniform lattice in G_γ (exercise). Thus,

$$\operatorname{tr} R(f) = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) I(\gamma, f).$$

Spectral side

Fact (*Gelfand*, *Graev* and *Piatetski-Shapiro*): $L^2(\Gamma\backslash G)$ decomposes discretely into a direct sum of irreducible representations of G, each occurring with finite multiplicity. Thus,

$$\operatorname{tr} R(f) = \sum_{\pi \in \hat{G}} m(\pi) \operatorname{tr} \pi(f)$$

where $\hat{G}$ is the unitary dual of G, $m(\pi)$ is the multiplicity of π in $L^2(\Gamma\backslash G)$ and $\operatorname{tr} \pi(f)$ is the trace of the operator $\int_G f(x)\pi(x)\, dx$ in the space of π.

Comparing the two we obtain the trace formula equality

$$\boxed{\sum_{\{\gamma\}} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) I(\gamma, f) = \sum_{\pi \in \hat{G}} m(\pi) \operatorname{tr} \pi(f)}$$

underlying the duality between conjugacy classes and irreducible representations.

Note that in the left-hand (geometric) side the first factor depends on Γ but not on f while the second depends on f but not on Γ. Similarly for the right-hand (spectral) side.

Irreducible representations of $SL_2(\mathbb{R})$

The standard two-dimensional representation

Its symmetric n power Sym^n – the unique $(n+1)$-dimensional irreducible representation of G

Consider the representation of G on the space of (smooth) functions on $\mathbb{R}^2 \setminus \{0\}$ (with G acting on the right). This representation is far from irreducible. In fact, for any (not necessarily unitary) character χ of $\mathbb{R}^*$ we can consider the subspace π_χ of those functions such that $f((rx, ry)) = (\chi(r)|r|)^{-1} f((x, y))$ for all $r \in \mathbb{R}^*$. (The factor $|r|$ is a convenient normalization factor.)

Fact: The π_χ's completely describe the irreducible representations of G. The characters of $\mathbb{R}^*$ are of the form $\chi(r) = |r|^s$ or $|r|^s \operatorname{sgn} r$ where sgn denotes the signum function. Since π_χ and $\pi_{\chi^{-1}}$ have the same character it is enough to consider the case $\mathrm{Re}(s) \geq 0$. The representation π_χ is irreducible unless χ is one of the characters $\chi_n(r) = r^n \operatorname{sgn}(r)$, $n \in \mathbb{Z}$. For $\chi = \chi_n$, $n \geq 0$ the representation π_χ has length three (two if $n = 0$), namely it has a unique irreducible quotient (for $n \neq 0$), equivalent to Sym^{n-1}, and two inequivalent irreducible subrepresentations $\pi_n^\pm$.

Conversely, if π is an irreducible representations of G then exactly one of the following holds

1. π is of finite dimension n, in which case it is equivalent to the irreducible quotient Sym^{n-1} of π_{χ_n}, or,
2. π is equivalent to π_χ, $\chi \neq \chi_n$, $n \in \mathbb{Z}$; χ is uniquely determined up to taking inverse, or,
3. π is equivalent to $\pi_n^\pm$ for a unique $n \in \mathbb{Z}_{\geq 0}$ and sign $\pm$.

Note that the finite-dimensional representations of G (except for the 1-dimensional one) are not unitarizable. For any $s \in \mathbb{C}$ the representation $\pi_{|\cdot|^s}$ admits a unique irreducible subquotient π_s which has a vector fixed under $SO(2)$, and such a vector (called *spherical vector*) is unique up to scalar. In fact, $\pi_s = \pi_{|\cdot|^s}$ unless s is an odd integer, in which case $\pi_s \simeq \mathrm{Sym}^{|s|-1}$. Note that $\pi_s \simeq \pi_{s'}$ if and only if $s = \pm s'$. Moreover, π_s is unitarizable if and only if $s \in \mathrm{i}\mathbb{R} \cup [-1, 1]$. The representation $\pi_n^\pm$, $n \in \mathbb{Z}_{>0}$ are exactly the irreducible subrepresentations of $L^2(G)$. They are therefore unitarizable and constitute the *discrete series* of G. (The representations $\pi_0^\pm$ are sometimes called *limits of discrete series*. They are also unitarizable.)

To tie the representation theoretic context to our previous discussion note that

$$L^2(\Gamma\backslash\mathbb{H}) = L^2(\Gamma\backslash G/K) = L^2(\Gamma\backslash G)^K$$

Fact: eigenfunctions of Δ on X with eigenvalue $(1-s^2)/4$ correspond to isometric embeddings of π_s in $L^2(\Gamma\backslash G)$. (Δ corresponds to the action of the generator of the center of the universal enveloping algebra on the spherical vector). Thus the multiplicity of $(1-s^2)/4$ is exactly $m(\pi_s)$.
Suppose that $f \in C_c^\infty(G//K)$, that is f is bi-K-invariant. We can think of f as a K-invariant compactly supported function on $\mathbb{H}$. As such it depends only on $\rho(\cdot, \mathrm{i})$. On the spectral side $\operatorname{tr}\pi(f)$ is non-zero only if $\pi = \pi_s$ for some $s \in \mathbb{C}$. In fact, $\operatorname{tr}\pi(f)$ is simply the scalar on which $\pi(f)$ acts on the spherical vector.

Let $A = \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t > 0\}$
$N = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\}$
$G = NAK$ (Gram-Schmidt).
We define the Abel transform of f by

$$\mathcal{A}(f)(t) = \int_{\mathbb{R}} f(\begin{pmatrix} e^{t/2} & x \\ 0 & e^{-t/2} \end{pmatrix})\, dx.$$

By a change of variable

$$\mathcal{A}(f)(2t) = |e^t - e^{-t}| \int_N f(n^{-1} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} n)\, dn$$

$$= |e^t - e^{-t}| \int_{T \backslash G} f(g^{-1} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} g)\, dg = |e^t - e^{-t}| I(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, f).$$

Fact: $\mathcal{A}$ is an algebra isomorphism between $C_c^\infty(G//K)$ and $C_c^\infty(\mathbb{R})^{even}$.
Plancherel inversion $f(e) = \int_{\mathbb{R}} \widehat{\mathcal{A}f}(r) r \tanh r \; dr$.
Moreover, it is easy to see from the polar decomposition $G = KAK$ that $\widehat{\mathcal{A}f}(r) = \operatorname{tr} \pi_{2\mathrm{i}r}(f)$.

Finally, the hyperbolic conjugacy classes in Γ correspond to the closed geodesics in $\Gamma\backslash X$ (not necessarily primitive). If γ is conjugate to $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$, $t > 0$ then the length $l(\gamma)$ of the corresponding closed geodesic is t. Moreover, $\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) = l(\gamma_0)$ where $\gamma = \gamma_0^k$, $\gamma_0 \in \Gamma$ and k is maximal with respect to this property (i.e., γ_0 is primitive). Altogether, in the co-compact case (assuming also Γ torsion free) Selberg's trace formula takes the form

$$\sum \hat{h}(r_n) = \frac{\mathrm{Area}(\Gamma\backslash\mathbb{H})}{2\pi} \int_{\mathbb{R}} \hat{h}(r) r \tanh(\pi r)\, dr + \sum \frac{l(\gamma_0) h(l(\gamma))}{e^{l(\gamma)/2} - e^{-l(\gamma)/2}}$$

where on the left-hand side $\frac{1}{4} + r_n^2$ range over the eigenvalues of the Laplacian while on the right-hand side γ range over the non-trivial conjugacy classes of Γ (i.e. closed geodesics) and γ_0 is as before.

Adèlic formulation

When dealing with congruence subgroups and Hecke operators it is useful to pass to adèlic setup. For instance

$$\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}) = \lim_{N} \Gamma_N \backslash \mathrm{SL}_2(\mathbb{R}).$$

Let R be the right regular representation of $G(\mathbb{A})$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Let $f \in C_c^\infty(G(\mathbb{A}))$. That is, f is compactly supported, smooth in the Archimedean variable and bi-invariant under an open subgroup of $G(\mathbb{A}_f)$ where $\mathbb{A}_f$ is the finite adeles. The operator $R(f)$ is integral with kernel

$$K_f(x, y) = \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma y).$$

Spectral theory for $GL(2)$

$$K_f(x,y) = K_f^{\text{disc}}(x,y) + K_f^{\text{cont}}(x,y)$$

where

$$K_f^{\text{disc}} = \sum_{\{\varphi\}} [R(f)\varphi](x)\overline{\varphi(y)},$$

sum over an ortho. basis of the discrete part (the sum of the irreducible subrepresentations) of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$, and

$$K_f^{cont}(x,y) = \sum_{\chi} \sum_{\{\varphi\}} \int_{-\infty}^{\infty} E(x, I(f,\chi,\mathrm{i}t)\varphi, \mathrm{i}t)\overline{E(y,\varphi,\mathrm{i}t)}\, dt$$

where the sum is over pairs $\chi = (\chi_1, \chi_2)$ of Dirichlet characters and over an ortho. basis $\{\varphi\}$ of the space

$$I(\chi) = \{\varphi : G(\mathbb{A}) \to \mathbb{C} | \varphi(\left(\begin{smallmatrix} t_1 & * \\ 0 & t_2 \end{smallmatrix}\right) g) = \chi_1(t_1)\chi_2(t_2)|\frac{t_1}{t_2}|^{\frac{1}{2}}\varphi(g)\}$$

with the inner product

$$(\varphi_1, \varphi_2) = \int_{\{\left(\begin{smallmatrix} at_1 & * \\ 0 & at_2 \end{smallmatrix}\right) : a \in \mathbb{R}_{>0}, t_1, t_2 \in \mathbb{Q}\} \backslash G(\mathbb{A})} \varphi_1(g) \overline{\varphi_2(g)} \, dg.$$

On this space there is a family of representations $I(\chi, s)$ given by

$$I(g, \chi, s)\varphi(\cdot) = (\varphi_s(\cdot g))_{-s}$$

where φ_s is defined by

$$\varphi_s(\left(\begin{smallmatrix} t & * \\ 0 & 1 \end{smallmatrix}\right) k) = |t|^s \varphi(k).$$

The (adelic) Eisenstein series $E(\varphi, s)$ is defined as the meromorphic continuation of the sum

$$E(g, \varphi, s) = \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} \varphi_s(\gamma g)$$

which converges for $\mathrm{Re}(s) > 1$. Whenever regular it defines an intertwining map from $I(\chi, s)$ to the space of automorphic forms on G.

The Selberg trace formula for $GL(2)/\mathbb{Q}$ revisited

Notation: $G = GL(2)$ over $\mathbb{Q}$, $Z =$ center, $T =$ diagonal torus, $B = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\}$ standard Borel,

$$I(s) = \operatorname{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} L^2(T(\mathbb{Q})\backslash T(\mathbb{A})^1, s) \quad s \in \mathbb{C}$$

induced representations

$$M(s) : I(s) \to I(-s)$$

intertwining operators given by meromorphic continuation of

$$[M(s)\varphi]_{-s}(g) = \int_{U(\mathbb{A})} \varphi_s(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) ug) \, du$$

and unitary for $s \in \mathrm{i}\mathbb{R}$.
Fix a level K and let $S \supseteq \{\infty\}$ be such that $K \supseteq \mathrm{GL}_2(\mathbb{Z}_p)$ $\forall p \notin S$.

Define

$$v_\gamma = \mathrm{vol}(G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A})^1) \quad \gamma \in G(\mathbb{Q}) \text{ elliptic},$$
$$v_T = \mathrm{vol}(T(\mathbb{Q})\backslash T(\mathbb{A})^1).$$

Also define

$$\lambda_{t,S} = -\sum_{p \notin S : |t_1|_p = |t_2|_p} \frac{1 - |1 - \frac{t_2}{t_1}|_p}{p-1} \log p$$

for regular $t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right) \in T(\mathbb{Q})$ and

$$\lambda_{t,S} = \lim_{X \mapsto \infty} \left[\frac{\sum_{1 \le n \le X : (n,S)=1} \frac{1}{n}}{\mathrm{res}_{s=1}\, \zeta^S(s)} - \log X \right] = \frac{\left[(s-1)\zeta^S(s)\right]'_{s=1}}{\mathrm{res}_{s=1}\, \zeta^S(s)}, \quad t \in Z(\mathbb{Q}).$$

For $u = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in U(\mathbb{A})$ set

$$\omega(t,u) = \sum_{p \le \infty} \omega_p(t,u)$$

where

$$\omega_p(t,u) = \begin{cases} \log \sqrt{|1 - \frac{t_2}{t_1}|_\infty^2 + |x|_\infty^2} & p = \infty \\ \log \max(|1 - \frac{t_2}{t_1}|_p, |x|_p) & p \in S \text{ finite} \\ \log \max(|1 - \frac{t_2}{t_1}|_p, |x|_p) & p \notin S, |t_1|_p \ne |t_2|_p \\ 0 & p \notin S, |x|_p \le 1 \\ \frac{1 - |1 - \frac{t_2}{t_1}|_p}{p-1} \log p + \log |x|_p & \text{otherwise} \end{cases}$$

Note that for almost all $u \in U(\mathbb{A})$ (namely, when $x_p \ne 0$ for all $p \in S$) $\omega(\cdot, u)$ is a continuous function on $T(\mathbb{A})$.

The trace formula identity

For any $f \in C_c^\infty(G(\mathbb{A})^1/K)$

$$\sum_{\gamma \text{ ell. conj. cls}} v_\gamma \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\, dx$$

$$+ v_T \sum_{t \in T(\mathbb{Q})} \int_{\mathbf{K}} \int_{U(\mathbb{A})} f(k^{-1}tuk)\omega(t,u)\, du\, dk$$

$$+ v_T \sum_{t \in T(\mathbb{Q})} \lambda_{t,S} \int_{\mathbf{K}} \int_{U(\mathbb{A})} f(k^{-1}tuk)\, du\, dk$$

$$= \operatorname{tr} R_{\mathrm{disc}}(f) - \frac{1}{4\pi} \int_{\mathrm{i}\mathbb{R}} \operatorname{tr}(M^{-1}(s)M'(s)I(f,s))\, ds$$

$$+ \frac{1}{4} \operatorname{tr}(M(0)I(f,0)).$$

Remarks

- In the case of a congruence subgroup $\Gamma = \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{GL}_2(\mathbb{R}) \cdot K$, taking $f = f_\infty \otimes \mathbf{1}_K$ reduces to the trace formula for $\Gamma \backslash \mathrm{GL}_2(\mathbb{R})$ with test function f_∞. The latter case simplifies since no regular $\mathbb{Q}$-hyperbolic orbital integrals (weighted or unweighted) show up.
- The adèlic setup is useful for taking traces of Hecke opertaors – they are incorporated by taking test functions at the p-adic places.
- The presentation above differs slightly from the "orthodox" one. It is suitable for more general (not necessarily compactly supported) test functions.
- Suppose that $f = \otimes f_p$ and for two distinct primes p_1, p_2 the regular Q_{p_i}-hyperbolic orbital integrals over f_{p_i} vanish. Then the trace formula simplifies to

$$\sum_{\gamma \text{ ell. conj. cls}} v_\gamma \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(x^{-1}\gamma x)\, dx = \operatorname{tr} R_{\mathrm{disc}}(f).$$

The Jacquet-Langlands correspondence

Archimedean case: Recall that the irreducible representations of the compact group $SO(3)$ are determined (up to equivalence) by their dimension, which can be any odd positive integer. We write σ_n for the n-dimensional irreducible representation of $SO(3)$ (n odd). Similarly, the irreducible square-integrable representations of $PGL(2,\mathbb{R})$ are indexed by odd positive integers. We write them as π_n (e.g., π_1 is the Steinberg representation). Writing χ_π for the character of a representation (a class function) we have

$$\chi_{\pi_n}\left(\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)\right) = -\chi_{\sigma_n}\begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \theta \in [0, 2\pi].$$

In fact, π_n together with the n-dimensional representation of $PGL(2,\mathbb{R})$ comprise the irreducible subquotients of an induced representation, whose character vanishes on the elliptic elements.

p-adic case

Fact: any p-adic field F admits a unique quaternion algebra D with center F. The multiplicative group $D^* = D \setminus \{0\}$ of D is an inner form of $G = GL(2, F)$.

$$\text{conjugacy classes of } D^* \longleftrightarrow \text{ elliptic conjugacy classes of } G.$$

$$x \leftrightarrow g \iff \operatorname{Trd} x = \operatorname{Tr} g \text{ and } \operatorname{Nrd} x = \det g$$

Theorem (local Jacquet-Langlands correspondence) There is a unique bijection

$$\text{(classes of) irred. rep'ns of } D^* \longleftrightarrow \text{(classes of) sqr.-int. rep'ns of } G.$$

determined by

$$\sigma \leftrightarrow \pi \iff \chi_\sigma(x) = -\chi_\pi(g)\ \forall x \leftrightarrow g$$

Note that finite-dimensional representations correspond to infinite-dimensional ones!

Global Jacuqet-Langlands correspondence

Fact The quaternion algebras over $\mathbb{Q} \longleftrightarrow$ finite subsets of places of $\mathbb{Q}$ (including the Archimedean one) with even cardinality.

$$D \mapsto S_D = \{p : D \otimes_{\mathbb{Q}} \mathbb{Q}_p \text{ is a quaternion algebra}\}$$

Given D let $G' = D^*$, $G'(\mathbb{A}) = (D \otimes_{\mathbb{Q}} \mathbb{A})^*$ and $G'(\mathbb{A})^1 = \{x \in G'(\mathbb{A}) : |\mathrm{Nrd}(x)| = 1\}$; $G' \backslash G'(\mathbb{A})^1$ is compact.

Theorem (global Jacquet-Langlands)

1. $L^2(G' \backslash G'(\mathbb{A})^1)$ is multiplicity free, i.e., all irreducible constituents occur with multiplicity one.
2. Suppose that $\sigma = \otimes_v \sigma_v$ is an irreducible constituent which is not one-dimensional. Define $\pi = \otimes_v \pi_v$ where $\pi_v = \sigma_v$ if $v \notin S_D$ and $\pi_v \overset{JL}{\longleftrightarrow} \sigma_v$ if $v \in S_D$. Then π is a cuspidal representation of $G(\mathbb{A})$.
3. Conversely, any cuspidal representation $\pi = \otimes \pi_v$ of $G(\mathbb{A})$ such that π_v is square-integrable for all $v \in S_D$ is obtained from an automorphic representation of G' by the above procedure.

Matching

Locally, to any $f' \in C_c^\infty(G')$ there exists $f \in C_c^\infty(G)$ such that $f' \leftrightarrow f$, i.e.:

$$I(g, f) = \begin{cases} I(x, f') & \text{if } x \leftrightarrow g, \\ 0 & \text{if } g \text{ is hyperbolic (or unipotent).} \end{cases}$$

This yields transfer of distributions from G' to G.
Local J.-L. reformulated: $\forall \sigma \in \hat{G}'$ the transfer of χ_σ is minus the character of some $\pi \in \hat{G}_{\text{sq.-int.}}$.

Idea of proof

Compare the trace formula for the two groups G' and G in the geometric side to get a spectral comparison.

Fact: conjugacy classes of $G'(F) \longleftrightarrow$ conjugacy classes of $G(F)$ which are elliptic for all $v \in S$.

Given $f' = \otimes f'_p$ choose $f = \otimes f_p$ as follows. For $p \notin S$ identify $G(\mathbb{Q}_p)$ with $D(\mathbb{Q}_p)$ (this is determined up to conjugation) and take $f'_p = f_p$. For $p \in S$ we take $f'_p \leftrightarrow f_p$. For this f

$$\operatorname{tr} R_G(f) = \sum_{\pi \text{ cuspidal}} \operatorname{tr} \pi(f) = \sum_{\gamma \in G(\mathbb{Q})_{\text{ell}}} I(\gamma, f)$$

Using the geometric side

$$\operatorname{tr} R_{G'}(f') = \sum_{\gamma \in G'(\mathbb{Q})} I(\gamma, f') = \sum_{\gamma \in G(\mathbb{Q})} I(\gamma, f) = \operatorname{tr} R_G(f).$$

We infer

$$\sum_{\pi'} m'(\pi') \operatorname{tr} \pi'(f) = \sum_{\pi} \operatorname{tr} \pi(f).$$

Remark

We cheated a little by assuming that $\mathrm{vol}(G'\backslash G'(\mathbb{A})^1) = \mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$. In fact, the argument can be used to prove this equality as well.
A curious outcome of the Jacquet-Langlands correspondence, due to Vigneras, is the existence of non-isometric compact hyperbolic surfaces with the same Laplacian spectrum! (cf. M. Kac: can you hear the shape of a drum?)

Artin's conjecture

Let $\rho : \Gamma_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible Galois representation.
Strong Artin Conjecture: ρ is modular, i.e., there exists a cuspidal representation $\pi = \otimes \pi_p$ of $\mathrm{GL}_n(\mathbb{A})$ (necessarily unique, denoted $\pi(\rho)$) such that whenever ρ is unramified at p, with $\rho(\mathrm{Frob}_p) \sim \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ we have

$$\pi_p = \mathrm{Ind}\left(\begin{pmatrix} t_1 & \ldots & * \\ & \ddots & * \\ & & t_n \end{pmatrix} \mapsto \alpha_1^{v_p(t_1)} \cdots \alpha_n^{v_p(t_n)}\right).$$

(Henceforth we write $g(\pi_p) = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ in this case.)
Thus, $L^S(s, \pi) = L^S(s, \rho)$ and in particular, the Artin conjecture holds for ρ.

What is known?

First note that if $\pi = \pi(\rho)$ exists then ω_π (central character of π) will correspond to $\det \rho$ under Class Field Theory.

$n = 2$

Deligne-Serre: if π is cuspidal and $\pi_\infty = \mathrm{Ind}\left(\left(\begin{smallmatrix} t_1 & * \\ 0 & t_2 \end{smallmatrix}\right) \to \mathrm{sgn}(t_1)\right)$ (i.e. π corresponds to a weight one modular form) then $\pi = \pi(\rho)$ for some odd Galois representation ρ (i.e. $\det \rho(c) = -1$ where c $=$ complex conjugation).

Khare-Wintenberger: conversely, any odd Galois representation is modular.

We can perform linear algebra operations and restriction/induction on Galois representations. Langlands' idea: there should be similar operations at the level of automorphic representations!

For instance, direct sum corresponds to forming Eisenstein series.

Denote by $\mathcal{A}(N)$ the set of automorphic (induced from cuspidal) representations of $GL_N(\mathbb{A})$.

Langlands functoriality conjecture Given $r : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(N, \mathbb{C})$ and $\pi \in \mathcal{A}(n)$ then there exists $\Pi \in \mathcal{A}(N)$ (denoted $\Pi = r(\pi)$) such that $g(\Pi_p) = r(g(\pi_p))$ for almost all p.
This is extremely deep and wide open. Among other things it would imply the Ramanujan conjecture in its most general form.
Gelbart-Jacquet: Functorial transfer exists for $\mathrm{Ad} : \mathrm{GL}(2, \mathbb{C}) \to \mathrm{GL}(3, \mathbb{C})$. Moreover, if π is cuspidal then $\mathrm{Ad}(\pi)$ is cuspidal unless π is dihedral.
It is useful to consider number fields other than $\mathbb{Q}$ (and their completions).
Let E/F be an extension of degree d.
Another (conjectural) instance of Langlands functoriality: $\forall \pi \in \mathcal{A}_F(n)$ there exists $\Pi = \mathrm{BC}_F^E(\pi) \in \mathcal{A}_E(n)$ such that $g(\Pi_w) = g(\pi_v)^{d(w)}$ for any unramified place w of E dividing v with $d(w) = [E_w : F_v]$.

Automorphic induction conjecture: $\forall \pi \in \mathcal{A}_E(n)$ there exists $\Pi = \mathrm{AI}_E^F(\pi) \in \mathcal{A}_F(nd)$ such that the eigenvalues of $g(\Pi_v)$ consist of the union over $w|v$ of the $d(w)$-th roots of the eigenvalues of $g(\pi_w)$. (Recall that $\sum_{w|v} d(w) = [E : F]$.)

Theorem (Saito-Shintani, Langlands, $n = 2$; Arthur-Clozel, $n > 2$) Base change and Automorphic induction exist for cyclic extensions. Moreover

1. if π is cuspidal then $\mathrm{BC}_F^E(\pi)$ is cuspidal unless there exists a non-trivial character ω of $I_F/F^* \operatorname{Nm} I_E$ such that $\pi = \pi \otimes \omega$.
2. $\mathrm{BC}_F^E(\pi) = \mathrm{BC}_F^E(\pi')$ if and only if $\pi' = \pi \otimes \omega$ for some character ω of $I_F/F^* \operatorname{Nm} I_E$.
3. Π is in the image of base change if and only if $\Pi^\sigma = \Pi$ for all $\sigma \in \mathrm{Gal}(E/F)$.
4. if $\rho : \Gamma_F \to \mathrm{GL}_n(\mathbb{C})$ and $\pi = \pi(\rho)$ then $\mathrm{BC}_F^E(\pi) = \pi(\rho|_{\Gamma_E})$.
5. if $\rho : \Gamma_E \to \mathrm{GL}_n(\mathbb{C})$ and $\pi = \pi(\rho)$ then $\mathrm{AI}_E^F(\pi) = \pi(\mathrm{Ind}_{\Gamma_E}^{\Gamma_F} \rho)$.

Special cases of the Artin conjecture

Suppose that $\rho : \Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ is irreducible. Consider the image in $\mathrm{SO}(3, \mathbb{C})$ under $\mathrm{Ad} : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_3(\mathbb{C})$. Any finite subgroup of $\mathrm{SO}(3, \mathbb{C})$ can be conjugated into $\mathrm{SO}(3, \mathbb{R})$. It is either cyclic, dihedral or isomorphic to the symmetry group of the Platonic solids: Tetrahedron, Octahedral (dual: cube), Dodecahedron (dual: Icosahedron) (symmetric group: A_4, S_4, A_5).
The cyclic case is ruled out since ρ is irreducible.
Dihedral case: $\rho = \mathrm{Ind}_{\Gamma_E}^{\Gamma_{\mathbb{Q}}} \theta$ for some quadratic extension E of $\mathbb{Q}$ and a character θ of Γ_E. In this case $\pi(\rho) = \mathrm{AI}_E^{\mathbb{Q}} \theta$ exists (where we view θ as a Hecke character of I_E).
Langlands: The strong Artin Conjecture holds in the tetrahedral case.
Tunnell: The strong Artin Conjecture holds in the octahedral case.
The proofs work for any number field.
The even icosahedral case is still wide open.

Idea of proof in the tetrahedral case

We have an exact sequence

$$1 \to V \to A_4 \to \mathbb{Z}/3 \to 1$$

Thus, ρ defines a character of order three on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, hence a cyclic cubic extension E of $\mathbb{Q}$.
$\rho_E = \rho|_{\Gamma_E}$ is dihedral – therefore $\pi(\rho_E) \in \mathcal{A}_E(2)$ exists.
Since $\pi(\rho_E)^\sigma = \pi(\rho_E^\sigma) = \pi(\rho_E)$ for any $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ there exists $\pi \in \mathcal{A}_\mathbb{Q}(2)$ such that $\mathrm{BC}_\mathbb{Q}^E(\pi) = \pi_E$. π is not unique, but if we require that $\omega_\pi \leftrightarrow \det \rho$ then it is. ($\omega_\pi|_{I_E} \leftrightarrow \det \rho|_{\Gamma_E}$ so precisely one of π, $\pi \otimes \omega$, $\pi \otimes \omega^2$ has the correct central character, where ω is a Dirichlet character of order 3 defining E.)
Obviously, π is our candidate for $\pi(\rho)$. But how to prove it?
There is no problem for split p – there $g(\pi_p) \sim \rho(\mathrm{Frob}_p)$.
But what about the non-split ones?

We first claim that $\Pi_1 := \mathrm{Ad}(\pi) \simeq \Pi_2 := \pi(\mathrm{Ad}\,\rho)$. Note that $\pi(\mathrm{Ad}\,\rho) \in \mathcal{A}_{\mathbb{Q}}(3)$ exists since $\mathrm{Ad}\,\rho = \mathrm{Ind}_{\Gamma_E}^{\Gamma_{\mathbb{Q}}} \theta$ for any of the three non-trivial quadratic characters of Γ_E trivial on the kernel of $\mathrm{Ad}\,\rho$. By a result of Jacquet-Shalika, it suffices to check that $L(s, \Pi_1 \otimes \tilde{\Pi}_2)$ has a pole at $s = 1$, and therefore, that

$$g((\Pi_1)_p \otimes g((\tilde{\Pi}_2)_p) = g((\Pi_2)_p) \otimes g((\tilde{\Pi}_2)_p) \qquad (4)$$

for almost all p. ($\tilde{\Pi}$ = contragredient of Π.) This is clear if p splits in E (in which case $g(\pi_p) \sim \rho(\mathrm{Frob}_p)$ and therefore $g((\Pi_1)_p) \sim g((\Pi_2)_p)$).

Otherwise, the image of Frob_p in A_4 has order 3 so that $\rho(\mathrm{Frob}_p) \sim \left(\begin{smallmatrix} \alpha & \\ & \omega\alpha \end{smallmatrix}\right)$, $\omega = e^{2\pi i/3}$ for some α and

$$g((\Pi_2)_p) \sim \mathrm{Ad}\,\rho(\mathrm{Frob}_p) \sim \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}$$

On the other hand $g(\pi_v)^3 \sim \rho(\mathrm{Frob}_p)^3$ (central) and $\det g(\pi_v) = \det \rho(\mathrm{Frob}_p)$ implies that

$$g(\pi_v) \sim \begin{pmatrix} \alpha & \\ & \omega\alpha \end{pmatrix} \text{ or } \begin{pmatrix} \omega^2\alpha & \\ & \omega^2\alpha \end{pmatrix}.$$

Hence,

$$g((\Pi_1)_v) \simeq \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

In both cases (4) holds.
To conclude: $\pi \in \mathcal{A}_{\mathbb{Q}}(2)$ satisfies:

1. $\mathrm{BC}_{\mathbb{Q}}^{E}(\pi) = \pi(\rho_E)$
2. $\omega_\pi = \det \rho$
3. $\mathrm{Ad}\,\pi = \pi(\mathrm{Ad}\,\rho)$

End of proof

To conclude we have to show that $g(\pi_p) \sim \rho(\mathrm{Frob}_p)$ for almost all p. Again, this is clear if p splits in E. Otherwise, suppose $g(\pi_p) \sim \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right)$ and $\rho(\mathrm{Frob}_p) \sim \left(\begin{smallmatrix} \alpha & \\ & \beta \end{smallmatrix}\right)$ are unequal. Then after relabeling $a = \omega\alpha$, $b = \omega^2\beta$. But we also know that $\{a/b, 1, b/a\} = \{\alpha/\beta, 1, \beta/\alpha\}$. This implies that $\omega\beta/\alpha = a/b = \alpha/\beta$ so that $\beta/\alpha = \pm\omega$. If $\beta/\alpha = \omega$ then $a = \beta$ and $b = \alpha$. If $\beta/\alpha = -\omega$ then

$$\mathrm{Ad}\,\rho(\mathrm{Frob}_p) \sim \begin{pmatrix} -\omega^2 & & \\ & 1 & \\ & & -\omega \end{pmatrix}$$

an element of order 6 which does not exist in A_4 – contradiction!

Twisted trace formula and Base Change

Let $\sigma \in \mathrm{Gal}(E/F)$ be a generator.
twisted conjugation in $G = \mathrm{GL}_n(E)$: $g_1 \sim g_2 \iff \exists x \in G$ s.t. $g_2 = x^{-1} g_1 x^{\sigma}$.
The point of departure of the trace formula approach to base change:
semisimple twisted conjugacy classes in $G \longleftrightarrow$ semisimple conj. classes in $G' = \mathrm{GL}_n(F)$, $\qquad g \mapsto \mathrm{Nm}\, g = g g^{\sigma} \dots g^{\sigma^{d-1}}$
This leads the way to a comparison

$$\int_{\mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E)} K_f(x, x^{\sigma})\, dx = \int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_E)} K_{f'}(x', x')\, dx'$$

(regularized integrals) for pairs of test functions $f \in C_c^{\infty}(\mathrm{GL}_n(\mathbb{A}_E))$ $f' \in C_c^{\infty}(\mathrm{GL}_n(\mathbb{A}_F))$ which are matching in the sense that the twisted orbital integral of f coincides with the usual orbital integral of f' for the corresponding class.
The ensuing spectral identity gives rise to the transfer of representations. Of course this is easier said than done..

Odds and ends

The base change, and its companion – automorphic induction, have great many applications. For instance the Local Langlands Conjecture asserts a remarkable relation between the n-dimensional representations of Γ_F, (or rather, the Weil group of F) and the irreducible representations of $\mathrm{GL}_n(F)$, for a local field F. This was proved by Harris–Taylor and Henniart. Base change was a key ingredient in the proof.
Before that, the proof of Fermat's Last Theorem by Wiles and Taylor–Wiles used the above-mentioned result of Langlands–Tunnell to "jump-start" the modularity argument.
Finally, the work of Clozel–Harris–Shepherd-Barron–Taylor on the Sato–Tate conjecture uses base change in an indispensable way.