RELATION SPACES OF HYPERPLANE ARRANGEMENTS AND MODULES DEFINED BY GRAPHS OF FIBER ZONOTOPES

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Contents

1. Introduction

This paper deals with two related topics. First, we consider complexes graded by geometric lattices, and specifically an inductive construction of such complexes starting from a collection of injections $U_a \hookrightarrow U_0$ indexed by the atoms a of the lattice. We call the complexes obtained from this construction minimal complexes, and our main interest is in criteria for their exactness. Apart from the well-known Orlik-Solomon algebra [OS80], an important special case is provided by the relation complex of a hyperplane arrangement, which describes the linear relations between its defining linear functionals. We give a simple proof of the n-formality of restrictions of complex reflection arrangements, i.e. we show the exactness of their relation complexes. In fact, we prove a somewhat stronger result, and provide an explicit contracting homotopy that is compatible with the grading by the intersection lattice (in our terminology, we show that the relation complex is admissible). While n-formality was known for reflection arrangements and restrictions of Coxeter arrangements as a consequence of their freeness [BT94], the construction of an admissible homotopy seems to be a new result. We then consider in §3 a general multilinear algebra construction, which we call the generalized Orlik-Solomon algebra of a complex graded by a geometric lattice, and show that it preserves the admissibility of complexes. In particular, this allows us to establish the exactness of certain minimal complexes constructed from relation spaces, if the relation complex itself is admissible.

The second topic is the study of modules over polynomial rings defined by the canonically labeled graphs of fiber zonotopes. Here, we consider an analogue of

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an exact sequence of Bernstein-Lunts [BL94, 15.7, 15.8] and Brion [Bri97, p. 12] for the algebra of piecewise polynomial functions on a complete simplicial fan, which relates this algebra to the polynomial functions on all cones. We consider fans obtained from hyperplane arrangements in a real vector space U^*. In the dual picture, these correspond to zonotopes (Minkowski sums of line segments) in the dual space U. In a first step, we derive a criterion for the existence of an exact sequence of Bernstein-Lunts type for modules defined in a purely combinatorial way by the labeled graph of a zonotope. Using the results of the first part, we then apply this criterion to the projected arrangements A_P governing the combinatorics of intersections of an arrangement A of rank n with the parallel translates of a fixed linear subspace $P^\perp \subset U^*$ of dimension k (which is assumed to satisfy a non-degeneracy condition). Dually, we pass from a zonotope $\mathcal{Z}$ dual to A to the fiber polytope $\mathcal{Z}_P$ in the sense of [BS92] of its projection in direction $P \subset U$. We use the 1-skeleton of $\mathcal{Z}_P$ and the relation complex of A to define a certain algebra $\mathcal{T}$ over the symmetric algebra R of the rank k space of the relation complex. Our main result is the existence of an exact sequence of Bernstein-Lunts type for $\mathcal{T}$, provided the relation complex of A is admissible (in particular if A is a restriction of a Coxeter arrangement). In the case $k = 0$ we recover the algebra of piecewise polynomial functions on the arrangement. (Note that restrictions of Coxeter arrangements are simplicial.) As a consequence, the algebra $\mathcal{T}$ is generated as an R-module by its homogeneous elements of degree at most $n - k$.

We note that the algebra $\mathcal{T}$ has in general a more complicated structure than the module of piecewise polynomial functions on a complete simplicial fan. For example, it is in general not free as a module over the polynomial ring R. In recent years, combinatorially defined algebras of the type we are considering have been studied by Guillemin and Zara [GZ99, GZ01, GZ03], but their focus is different, since they are interested in cases where the algebra in question is actually free over the underlying polynomial ring.

It is natural to ask if our result extends to arbitrary simplicial hyperplane arrangements. Although it is easy to see that this is true for ranks up to four, the general case remains open.

In the paper [FLM] our result is applied to derive an algebraic formula evaluating certain multidimensional limits, which is then used to analyze the spectral side of Arthur's trace formula. We hope that the combinatorial material presented in this paper is also of some independent interest.

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2. Minimal complexes graded by geometric lattices

We start with a simple general construction of complexes graded by geometric lattices which encompasses previously known constructions. It is modeled on the relation complexes of [BT94] and the minimal complexes of [BL94, Ch. 15].

For any ranked poset $(L, \leq)$ with rank function rk we write $x \prec y$, or equivalently $y \succ x$, if y covers x, i.e. if $x < y$ and $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$. Let now L be a geometric lattice of rank n with minimal element 0. Set

$$L_i = \{x \in L : \mathrm{rk}(x) = i\}, \quad i = 0, \ldots, n,$$

and denote by $\mathbb{A}(L) = L_1$ the set of atoms of L. By abuse of notation we will denote the maximal element of L by L as well. For any $x \in L$ the lower interval $L_{\leq x} = \{y \in L : y \leq x\}$ is also a geometric lattice, whose atoms are $\mathbb{A}_{\leq x} = \{a \in \mathbb{A} : a \leq x\}$.

Definition 1. Let R be a ring. A *compatible L-grading* on a chain complex

$$0 \to V_n \to \ldots V_i \xrightarrow{\partial_i} V_{i-1} \cdots \to V_0 \tag{1}$$

of R-modules is given by submodules V_x of $V_{\mathrm{rk}(x)}$, $x \in L$, such that

$$V_i = \oplus_{\mathrm{rk}(x)=i} V_x, \quad i = 0, \ldots, n,$$

and

$$\partial_{\mathrm{rk}(x)}(V_x) \subseteq V_{\prec x} := \oplus_{y \prec x} V_y \quad x \in L \setminus \{0\}.$$

Note that the notation V_0 is unambiguous. If V is a chain complex with a compatible L-grading, then for any $x \in L$ the complex $V_{\leq x} := \oplus_{y \leq x} V_y$ inherits a compatible grading by $L_{\leq x}$.

Definition 2. An *atomic datum* $\mathcal{D}$ (over R) with respect to L consists of an R-module U_0 together with submodules U_a for each atom a of L. We set $U^x := \sum_{a \leq x} U_a$ for $x \in L$. An atomic datum is called *essential* if $U_0 = U^L$.

Given an atomic datum $\mathcal{D} = (U_0, (U_a))$ there exists a unique chain complex (1) endowed with a compatible L-grading and satisfying the following properties:

1. $V_0 = U_0$,
2. for all $a \in \mathbb{A}(L)$ we have $V_a = U_a$ and $\partial_1|_{V_a}$ is the inclusion $U_a \hookrightarrow U_0$,
3. for all $i > 1$ and $x \in L_i$ we have

   $$V_x = \mathrm{Ker}\, \partial_{i-1}|_{V_{\prec x}}$$

   and $\partial_i|_{V_x}$ is the inclusion $V_x \hookrightarrow V_{\prec x}$.

We call $V = \oplus_{x \in L} V_x$ the *minimal complex* of the atomic datum $\mathcal{D}$, and denote it by $\mathcal{O}(\mathcal{D}) = \mathcal{O}(\mathcal{D}, L)$.

Remark 1. Suppose that the complex (1) admits a compatible L-grading and that $\partial_1|_{V_a}$ is injective for each $a \in \mathbb{A}(L)$. Assume in addition that for all $x \in L$ the subcomplex $V_{\leq x}$ is acyclic (i.e. $\mathrm{Ker}\, \partial_i = \mathrm{Im}\, \partial_{i+1}$, $i = 1, \ldots, n$, but we do not require that ∂_1 is onto). Then by the uniqueness of the minimal complex it follows that V is isomorphic to $\mathcal{O}(V_0, (V_a))$ as an L-graded module.

The basic example is when $U_a = U_0$ for all $a \in \mathbb{A}(L)$. This is called the atomic datum with constant coefficients. If $R = U_0 = \mathbb{Z}$ then as an L-graded chain complex the minimal complex is the usual Orlik-Solomon algebra of L with coefficients in $\mathbb{Z}$ (cf. [OS80]).

Another important example is when R is a field k, and L is the intersection lattice of a hyperplane arrangement A in a vector space U^* given by hyperplanes $H_a = \{x \in U^* : \langle x, u_a \rangle = 0\}$ indexed by the atoms of L. The atomic datum pertaining to A is by definition $(U, (H_a^{\perp})) = (U, (ku_a))$. We call the resulting minimal complex the *relation complex* of A. Its graded pieces are the relation spaces considered in [BT94]. In particular, an essential hyperplane arrangement is called k-formal, if the first $k-1$ homology groups of (1) vanish, i.e. if $\operatorname{Ker} \partial_i = \operatorname{Im} \partial_{i+1}$, $i = 1, \dots, k-1$. For example, 2-formality means that the linear dependencies among the linear functionals u_a, $a \in \mathbb{A}$, are generated by those induced by rank two elements of the intersection lattice (or, what amounts to the same, by the linear dependencies among three functionals); 3-formality means that in addition the relations among the relations are generated by those induced by rank three elements and so forth. In particular, n-formality means that (1) is acyclic. (Note that ∂_n is always injective.) The relation spaces are the degree 0 part (with respect to the usual grading on $\operatorname{Sym} U^*$) of $\mathcal{O}(\operatorname{Sym} U^* \otimes U, (\operatorname{Sym} H_a \otimes H_a^{\perp}))$. The main result of [BT94] is that the latter is acyclic if A is a free arrangement.

Definition 3. Let L be a geometric lattice. An element $x \in L$ is called *reducible*, if there exist $y, z \in L \setminus \{0\}$ such that the join gives an isomorphism of $L_{\leq y} \times L_{\leq z}$ with $L_{\leq x}$. In this case we write $x = y \varovee z$. Otherwise x is called *irreducible.* The set of irreducible elements of L is denoted by L_{irr}.

Any $x \in L$ can be written uniquely as $\varovee_{i=1}^m x_i$ where x_i are irreducible. In particular, $L_{\leq x} = L_{\leq x_1} \times \cdots \times L_{\leq x_m}$. Note that if $x, y \in L_{irr}$ are not disjoint (that is, if $x \wedge y \neq 0$) then $x \vee y \in L_{irr}$.

Lemma 1. *Suppose that $\mathcal{D}$ is the atomic datum of a hyperplane arrangement with intersection lattice L. Then for any reducible $x \in L$ we have $\mathcal{O}(\mathcal{D}, L)_x = 0$.*

Proof. We will prove the statement by induction on $\operatorname{rk}(x)$. Suppose that $x = y \varovee z$. Then $L_{\prec x} = (y \varovee L_{\prec z}) \cup (L_{\prec y} \varovee z)$. If neither y nor z is an atom then by induction hypothesis $\mathcal{O}(\mathcal{D})_{\prec x} = 0$, and therefore $\mathcal{O}(\mathcal{D})_x = 0$. Suppose that z is an atom. Then $L_{\prec x} = \{y\} \cup (L_{\prec y} \varovee z)$. If y is not an atom then by induction hypothesis $\mathcal{O}(\mathcal{D})_{\prec x} = \mathcal{O}(\mathcal{D})_y$ and therefore $\mathcal{O}(\mathcal{D})_x = 0$ since ∂ is injective on $\mathcal{O}(\mathcal{D})_y$. Finally, it remains to consider the case where both y and z are atoms. In this case $L_{\prec x} = \{y, z\}$ and $\mathcal{O}(\mathcal{D})_x = 0$ since $U_y \cap U_z = 0$. □

Our motivating problem is to provide sufficient conditions for the acyclicity of $\mathcal{O}(\mathcal{D})$. It will be useful to introduce a stronger notion.

Definition 4. Let (1) be a chain complex with compatible L-grading. An *admissible homotopy* for V is a contracting homotopy d for ∂, i.e.

$$d_{i-1}\partial_i + \partial_{i+1} d_i = \mathrm{id}, \quad i = 1, \dots, n,$$

where we only require that d_0 be defined on $\operatorname{Im}\partial_1$, with the property $d_{\mathrm{rk}(x)}(V_x) \subseteq V_{\succ x} = \oplus_{y\succ x} V_y$ for all $x \in L \setminus \{0\}$. We say that V is *admissible* if for any $x \in L$ the complex $V_{\leq x}$ admits an admissible homotopy. Analogously, we say that an atomic datum $\mathcal{D}$ is admissible if $\mathcal{O}(\mathcal{D}, L)$ is admissible.

Clearly, the product of admissible atomic data is admissible. Henceforth R will be assumed to be a field k of characteristic 0 or more generally an algebra over $\mathbb{Q}$.

We now describe a way to construct admissible homotopies.

Definition 5. Let $(U_0, (U_a)_{a\in\mathbb{A}})$ be an atomic datum. A morphism $d_0 : U^L \to \oplus_{a\in\mathbb{A}} U_a$ is called an *admissible section* if the following two conditions are satisfied:

1. For all $x \in L_{irr} \setminus \{0\}$ there exists a non-zero integer h^x such that the composition
$$U^x \xrightarrow{d_0} \oplus_{a\in\mathbb{A}} U_a \xrightarrow{\pi_{\leq x}} \oplus_{a\leq x} U_a \xrightarrow{\sum} U^x$$
is h^x times the identity map, where $\pi_{\leq x}$ is the projection.
2. If $a, a' \in \mathbb{A}$ and $a \vee a'$ is reducible, then the a'-component of $d_0|_{U_a}$ vanishes.

If d_0 is an admissible section for the atomic datum $(U_0, (U_a))$, we denote by d_0^x the restriction of $\pi_{\leq x} \circ d_0$ to U^x. Clearly, d_0^x is an admissible section for the atomic datum $(U_0, (U_a)_{a\leq x})$.

Remark 2. Suppose that d is an admissible section for an atomic datum $(U_0, (U_a))$. Let h^x, $x \in L_{irr}$, be as above. Then for all $x \leq z \in L_{irr}$ we have
$$\sum_{y\in L_{irr}: x\prec y\leq z} (h^y - h^x) = h^z - h^x.$$
Indeed, by restricting to $L_{\leq z}$ we can assume that z is the maximal element. Since
$$\{a \in \mathbb{A} : a \not\leq x\} = \coprod_{y\succ x} \{a \in \mathbb{A} : a \leq y, a \not\leq x\},$$
we have
$$d_0 - d_0^x = \sum_{y\succ x} (d_0^y - d_0^x)$$
on U^x. By the second condition on d_0 we can take the sum only over irreducible y. Applying ∂ to both sides we get the required equality.

Proposition 1. *Suppose that d is an admissible section for the atomic datum $(U_0, (U_a))$ of a hyperplane arrangement with intersection lattice L. Let h^x, $x \in L_{irr}$, be as above. Then for each $x \in L_{irr}$, $\frac{1}{h^x} d_0^x$ extends to an admissible homotopy on $\mathcal{O}(\mathcal{D}_{\leq x}, L_{\leq x})$.*

Proof. We will construct the d_i^z's, $i > 0$ by induction on i. Suppose that d_j^z are defined for all $j < i$ and $z \in L$, and satisfy
$$\partial_{j+1} d_j^z + d_{j-1}^z \partial_j = h^z \,\mathrm{id} \quad 0 < j < i, z \in L.$$

In particular,

$$\partial_i d^z_{i-1} \partial_i = h^z \partial_i. \tag{2}$$

The latter is also valid for $i = 1$ by the property of d_0. We define d^z_i by

$$d^z_i = \oplus_{x,y \in L_{irr}:\mathrm{rk}(x)=i, z \geq y \succ x} d_{x;y} \tag{3}$$

where $d_{x;y} : V_x \to V_y$ is such that

$$\partial_{i+1} d_{x;y} = h^y \,\mathrm{id} - d^y_{i-1} \partial_i.$$

This is well-defined because ∂_{i+1} is injective on each V_y and the image of $h^y \,\mathrm{id} - d^y_{i-1}\partial_i$ lies in $\mathrm{Ker}\, \partial_i$ by (2). For compatibility we write $d_{0;a} = d^a_0$ for any atom a so that (3) is satisfied for $i = 0$ as well. To show

$$\partial_{i+1} d^z_i + d^z_{i-1} \partial_i = h^z \,\mathrm{id} \quad z \in L$$

let $x \in L_i$ with $x \leq z$. Write $\partial_i = \oplus_{x' \prec x} \partial_{i,x'}$. Then on V_x we have

$$\partial_{i+1} d^z_i + d^z_{i-1} \partial_i = \sum_{y \in L_{irr}: z \geq y \succ x} (h^y \,\mathrm{id} - d^y_{i-1} \partial_i) + d^z_{i-1} \partial_i$$

$$= \sum_{y \in L_{irr}: z \geq y \succ x} \left[h^y \,\mathrm{id} - \sum_{x',y' \in L_{irr}: x' \prec x, y \geq y' \succ x'} d_{x';y'} \partial_{i,x'} \right] + \sum_{x',y' \in L_{irr}: x' \prec x, z \geq y' \succ x'} d_{x';y'} \partial_{i,x'}.$$

The contribution from any pair $y' \succ x'$ with $y' \neq x$ to the last sum cancels with its contribution to the middle sum for $y = x \vee y'$. (Note that if $y \notin L_{irr}$ then $i = 1$ by the remark before Lemma 1, and $d^{y'}_0|_{U_x} = 0$ by the condition on d_0.) Therefore, only the terms $y' = x$ contribute and we conclude that $\partial_{i+1} d^z_i + d^z_{i-1} \partial_i$ preserves V_x. Since ∂_i is injective on V_x it remains to show that

$$\partial_i (\partial_{i+1} d^z_i + d^z_{i-1} \partial_i) = h^z \partial_i$$

on V_x. This follows from (2). □

Let now G be an irreducible finite complex reflection group on a vector space V and consider the reflection arrangement $\{H_a\}_{a \in \mathbb{A}}$ consisting of the reflecting hyperplanes H_a of G (cf. [OT92, Ch. 6]). Let L be the corresponding intersection lattice. Note that L is irreducible. Choose a G-invariant scalar product $(\cdot,\cdot)$ on V. Let $\mathcal{D} = (V, (H_a^\perp))$ be the corresponding atomic datum (where we identified V and V^* through $(\cdot,\cdot)$). For each $a \in \mathbb{A}$ choose $w_a \in H_a^\perp$ such that $(w_a, w_a) = 1$.

Proposition 2. *Under these assumptions* $d_0(v) = (2(v, w_a)w_a)_{a \in \mathbb{A}}$ *is an admissible section.*

Proof. The first property of admissibility follows from [OT92, Proposition 6.93]. In fact $h^x = 2|\mathbb{A}_{\leq x}| / \mathrm{rk}(x)$. (If G is a real reflection group then h^x is the Coxeter number of the group generated by the reflections in x; cf. [ibid., 6.99].) More generally, if $x = \varowedge_{i=1}^{k} x_i$ is the decomposition of $x \in L$ into irreducibles, then $U^x = \oplus U^{x_i}$ is an orthogonal direct sum with respect to $(\cdot, \cdot)$ and therefore

$$\partial d_0^x(\sum_{i=1}^{k} v_i) = \sum_{i=1}^{k} h^{x_i} v_i, \quad v_i \in U^{x_i}.$$

The second property of admissibility follows from the fact that if $a \vee a'$ is reducible then w_a and $w_{a'}$ are orthogonal. $\square$

More generally, let $x \in L$ and proj_x be the orthogonal projection $V \to x = V/x^\perp$. Set

$$d_{0,\geq x}(v) = \left(\sum_{a' \in \mathbb{A}: a' \vee x = a} 2(v, w_{a'}) \operatorname{proj}_x(w_{a'}) \right)_{a \in \mathbb{A}(L_{\geq x})}, \quad v \in x.$$

We have $\partial \circ d_{0,\geq x} = \operatorname{proj}_x \partial \circ d_0|_x$.

Proposition 3. *$d_{0,\geq x}$ is an admissible section for the restricted hyperplane arrangement on x.*

Proof. Let $y \in L_{\geq x}$ and $y = \varowedge y_j$ its decomposition into irreducibles in L. Then for $v = \sum v_j \in U^y \cap x$, $v_j \in U^{y_j}$, we have

$$\partial d_{0,\geq x}^y(v) = \operatorname{proj}_x \partial d_0^y(v) = \sum h^{y_j} \operatorname{proj}_x v_j.$$

Suppose now that y is irreducible in $L_{\geq x}$. Then $y_j \leq x$ except for a single j, say $j = 1$. Therefore, if $v = \sum v_j \in U^y \cap x$, then $\partial d_{0,\geq x}^y(v) = h^{y_1} \operatorname{proj}_x v_1 = h^{y_1} v$.

To check the second condition of admissibility suppose that a_1, a_2 are distinct atoms in $L_{\geq x}$. Let $0 \neq v \in a_1^\perp \cap x$. Suppose that the a_2-component of $d_{0,\geq x}(v)$ is non-zero. Then $(v, w_z) \neq 0$ for some $z \in \mathbb{A}$ such that $z \vee x = a_2$. Thus, $v \notin H_z$ and if s_z is a reflection around H_z then modulo $x^\perp$ the elements v, $s_z(v)$ and w_z are linearly dependent, but pairwise non-proportional. Therefore $a_1 \vee a_2$ is irreducible in $L_{\geq x}$. $\square$

Since any finite complex reflection group is a product of irreducible ones, we proved:

Corollary 1. *Any restriction of a reflection arrangement is admissible.*

Remark 3. It was proved in [BT94] that every *free* hyperplane arrangement is n-formal. (Cf. [ibid.] and [OT92] for the definition and properties of free arrangements.) It is also known that restrictions of Coxeter arrangements are always free [OT93]. Therefore (1) is acyclic in this case. However, the argument in [BT94] is indirect and does not seem to produce a contracting homotopy (let alone an admissible one) explicitly. Also, the proof of [OT93] involves a case by case analysis to deal with restrictions to elements of rank bigger than one, which is avoided in the argument above.

It is instructive to explicate the relation complexes of the various root systems. We will only do this for rank two and for the infinite families of irreducible Coxeter arrangements.

Rank two cases. For any distinct atoms a, b define $\operatorname{proj}_{a,b} V \to V_a \oplus V_b$ by $\partial \operatorname{proj}_{a,b}(v) = v$. Then $d_0 = \sum_{a \neq b \in \mathbb{A}} \operatorname{proj}_{a,b}$ is an admissible section. Here $h = \binom{|A|}{2}$ and $h^a = |A| - 1$ for each atom.

A_n **case.** Let U be an $(n+1)$-dimensional vector space with basis $e_1, \dots, e_{n+1}$ and the standard scalar product and consider the (non-essential) reflection arrangement of type A_n given by the vectors $e_i - e_j$, $1 \le i < j \le n+1$. (Its restrictions are also of type A, so they do not need to be considered separately.) Then L is the partition lattice on $\{1, \dots, n+1\}$ with rank function $\operatorname{rk}((I_1, \dots, I_m)) = \sum_{j=1}^m (|I_j| - 1)$. Let V be the irrelevant ideal of the exterior algebra $\wedge^{\cdot} U$. As a vector space, V has a basis $e_I = e_{i_1} \cdots e_{i_k}$, $I = \{i_1, \dots, i_k\}$, $1 \le i_1 < \cdots < i_k \le n+1$, $k \ge 1$. We also set $e_\emptyset = 0$. We grade V by $\deg(e_I) = |I| - 1$. We have the differential

$$\partial(e_I) = \sum_{j=1}^{k} (-1)^{j+1} e_{I \setminus \{i_j\}}.$$

Then V is compatibly L-graded by

$$V_{(I_1, \dots, I_m)} = \begin{cases} U, & \text{if } (I_j) \text{ is the singleton partition,} \\ \mathbb{C} e_{I_j}, & \text{if there is a unique } j \text{ such that } |I_j| > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Multiplication by e_1 gives an admissible contracting homotopy for V and therefore V is the relation complex for A_n (cf. Remark 1). This homotopy is not equivariant with respect to the Weyl group $W = S_{n+1}$. The homotopy constructed above is $d_0(v) = \frac{1}{n+1} \sum_{i=1}^{n+1} e_i v$. Note that as a W-module,

$$V_k \simeq \wedge^{k+1}(\mathrm{St} \oplus \mathbb{C}) = \wedge^k \mathrm{St} \oplus \wedge^{k+1} \mathrm{St},$$

where St is the standard n-dimensional representation of W. Recall that the representations $\wedge^k \mathrm{St}$ are irreducible.

Other infinite families. The restrictions of the reflection arrangements of type B and D are isomorphic to the hyperplane arrangements $\Phi_{n,m}$, $m \le n$ given by the vectors $\{e_i \pm e_j : 1 \le i < j \le n\} \cup \{e_i : i \le m\}$ in an n-dimensional space with basis $e_1, \dots, e_n$ and the standard scalar product. For $m = n$ we obtain a reflection arrangement of type B_n, and for $m = 0$ an arrangement of type D_n. Here $L = L^{\Phi_{n,m}}$ is the lattice whose elements consist of the following data:

1. a (possibly empty) subset J of $\{1, \dots, n\}$. If $J = \{j\}$ is a singleton, then $j \le m$,
2. an unordered partition $(I_1, \dots, I_k)$ of $\{1, \dots, n\} \setminus J$,
3. for each $j = 1, \dots, k$ a class of functions $\epsilon : I_j \to \{\pm 1\}$, where we identify ϵ and $-\epsilon$.

The rank function is $|J| + \sum_{j=1}^k (|I_j| - 1)$. For any $I \subseteq \{1, \dots, n\}$ with $|I| > 1$ and a class of functions $\epsilon : I \to \{\pm 1\}$ let $x_{(I,[\epsilon])}$ be the element of L of rank $|I| - 1$ corresponding to $J = \emptyset$, the partition whose only subset of size > 1 is I, and the class of ϵ. Similarly, for any $J \subset \{1, \dots, n\}$ (possibly empty but with the restriction on singletons above) let $y_J \in L_{|J|}$ correspond to the singleton partition of $\{1, \dots, n\} \setminus J$. Let $L_{irr,A} = \{x_{(I,[\epsilon])}\}$ and $L_{irr,B} = \{y_J\}$.

Let W be the quotient of the vector space with basis

$$e_{(I,\epsilon)}, \quad \emptyset \neq I \subset \{1, \dots, n\}, \epsilon : I \to \{\pm 1\},$$

by $e_{(I,\epsilon)} + (-1)^{|I|} e_{(I,-\epsilon)}$. We grade W by $|I|$ and observe that (W, ∂) is a chain complex with respect to

$$\partial e_{(I=\{i_1,\dots,i_k\},\epsilon)} = \sum_{j=1}^{k} (-1)^{j+1} \epsilon(i_j) e_{(I\setminus\{i_j\},\epsilon|_{I\setminus\{i_j\}})}.$$

For any $\emptyset \neq J \subset \{1, \dots, n\}$ let W_J be the span of $e_{(J,\epsilon)}$ in $W_{|J|}$.

Let V^{B_n} be the mapping cone of the identity map on (W, ∂). That is $V_k^{B_n} = W_k \oplus W_{k+1}$, $k = 0, \dots, n$, with the differential

$$(w_k, w_{k+1}) \mapsto (-\partial_k w_k, w_k + \partial_{k+1} w_{k+1}).$$

V^{B_n} is compatibly L^{B_n}-graded by

$$\begin{gathered} V^{B_n}_{x_{(I,[\epsilon])}} = \{0\} \times \mathbb{C} e_{(I,\epsilon)} \subset V^{B_n}_{|I|-1}, \\ V^{B_n}_{y_J} = W_J \times \{0\} \subset V^{B_n}_{|J|}, \quad J \neq \emptyset, \\ V^{B_n}_0 = \{0\} \times W_1, \\ V^{B_n}_x = 0, \quad \text{if } x \notin L^{B_n}_{irr,A} \cup L^{B_n}_{irr,B}. \end{gathered}$$

The map $d(w_k, w_{k+1}) = (w_{k+1}, 0)$ is an admissible contracting homotopy for V^{B_n}.

For any $j > m$ let (U^j, ∂^j) be the chain complex with basis elements f^j_J, $j \notin J \subset \{1, \dots, n\}$, of degree $|J| + 1$ and differential

$$\partial^j f^j_{J=\{i_1,\dots,i_k\}} = \sum_{l=1}^{k} (-1)^{l+1} f^j_{J\setminus\{i_l\}}.$$

For any $k \neq j$ a homotopy $d^{U^j,k}$ on U^j is given by

$$f^j_J \mapsto \begin{cases} (-1)^{|\{x \in J : x < k\}|} f^j_{J\cup\{k\}}, & k \notin J, \\ 0, & \text{otherwise.} \end{cases}$$

We also set $d^{U^j,j} = 0$. Let $U := \oplus_{j>m} U^j$ with the differential $\partial^U = \oplus \partial^j$. Define $d^{U,k} = \oplus_{j>m} d^{U^j,k}$ and set

$$D^U = \begin{cases} \frac{1}{n-1} \sum_{k=1}^{n} d^{U,k}, & m = 0, \\ d^{U,1}, & \text{otherwise.} \end{cases}$$

Then D is a homotopy of U.

The map $\phi : W \to U$ given by

$$e_{(I,\epsilon)} \mapsto \prod_{i \in I} \epsilon(i) \sum_{x \in I, x > m} (-1)^{|\{i \in I : i > x\}|} \epsilon(x) f^x_{I\setminus\{x\}}$$

is easily seen to be a map of chain complexes. Let K be its kernel. Then we have an exact sequence

$$0 \to K \oplus W[1] \to V^{B_n} \xrightarrow{(w,w')\mapsto\phi(w)} U \to 0.$$

It follows that $K \oplus W[1]$ is exact and compatibly $L^{\Phi_{n,m}}$-graded. Therefore, $V^{\Phi_{n,m}} = K \oplus W[1]$.

Consider $d^{W,k} : W \to W$ given by

$$d^{W,k} e_{(I,\epsilon)} = \begin{cases} (-1)^{|\{x\in I : x<k\}|}\frac{1}{2}\left(e_{(I\cup\{k\},\epsilon\cup\{(k,1)\})} - e_{(I\cup\{k\},\epsilon\cup\{(k,-1)\})}\right), & k \notin I, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that $\phi d^{W,k} = d^{U,k}\phi$. In particular $d^{W,k}$ preserves K. Set

$$D^W = \begin{cases} \frac{1}{n-1}\sum_{k=1}^n d^{W,k}, & m = 0, \\ d^{W,1}, & \text{otherwise.} \end{cases}$$

Then $\phi D^W = D^U \phi$. The map

$$(w_k, w_{k+1}) \mapsto (-D^W_k w_k + w_{k+1} - \partial_{k+2} D^W_{k+1} w_{k+1} - D^W_k \partial_{k+1} w_{k+1}, D^W_{k+1} w_{k+1})$$

is a homotopy of $V^{\Phi_{n,m}}$. It is clearly admissible.

3. A generalization of the Orlik-Solomon algebra

In this section we consider a general construction, modeled after the Orlik-Solomon algebra, which allows to pass from an admissible complex of vector spaces (V_x) to an admissible complex of modules over the symmetric algebra $\mathrm{Sym}(V_0)$. We then apply this construction to truncations of a lattice L and more specifically to the relation complexes of §2.

Suppose we are given a chain complex $V = \oplus_{x\in L} V_x$ of vector spaces over a field k of characteristic zero with a compatible grading by a geometric lattice L. Let $S = S(V)$ be the universal supercommutative algebra generated by V, i.e. S is the quotient of the tensor algebra of V by the ideal generated by $uv - (-1)^{ij} vu$ for $u \in V_i$, $v \in V_j$. It is naturally graded by assigning V_i the degree i. By extending ∂ to a (super-)derivation $\tilde{\partial}$ on S, we obtain a differential graded algebra $(S, \tilde{\partial})$. Note that S_0 is the symmetric algebra of V and S is an algebra over S_0.

The algebra S carries a canonical grading by the lattice L in which $V_x \subset S_x$ and

$$a \in S_x, b \in S_y, x, y \in L \implies ab \in S_{x\vee y}.$$

Note that the notation S_0 is unambiguous and that for any $a \in \mathbb{A}(L)$ the component S_a is via $\tilde{\partial}$ isomorphic to the ideal of S_0 generated by $\partial(V_a) \subset V_0$.

This grading by L is however not compatible in the sense of Definition 1. The problem is that if $r_i \in V_{x_i}$, $i = 1, \dots, m$, with

$$\sum_{i=1}^m \mathrm{rk}(x_i) > \mathrm{rk}(\vee x_i), \tag{4}$$

the product $r_1 \cdots r_m$ does not appear in degree $\operatorname{rk}(\vee x_i)$. To rectify this, we say that $x_1, \ldots, x_m \in L$ are *dependent* if (4) holds and consider the vector space I spanned by all products

$$r_1 \cdots r_m, \quad r_i \in V_{x_i}, i = 1, \ldots, m, x_1, \ldots, x_m \in L \text{ dependent.}$$

Note that this extends the standard notion of dependency of atoms. Since a set containing a dependent set is again dependent, I is a graded ideal of S. It follows that $\mathcal{I} := \mathcal{I}(V) = I + \tilde{\partial}(I)$ is a differential ideal of S. We can now define the main object of this section.

Definition 6. The generalized Orlik-Solomon algebra of the L-graded complex V is the differential graded algebra $\mathcal{A} = \mathcal{A}(L, V) := S(V)/\mathcal{I}$.

Proposition 4. *The generalized Orlik-Solomon algebra $\mathcal{A}(L, V)$ is compatibly L-graded.*

Proof. Clearly, the ideal I is L-graded. Let $r_i \in V_{x_i}$, where $x_1, \ldots, x_m$ is a dependent set. Let $y_1, \ldots, y_m \in L$ and $1 \leq j \leq m$ be such that $y_i = x_i$ for $i \neq j$ and $y_j \prec x_j$. Then either $y_1, \ldots, y_m$ are dependent or $\vee y_i = \vee x_i$.

Therefore we can write

$$\tilde{\partial}(r_1 \cdots r_m) = u + v$$

where $u \in I$ and $v \in S_{\vee x_i}$. Thus, $v \in \mathcal{I}_{\vee x_i}$. It follows that $\mathcal{I}$ is L-graded and for all $x \in L$ we have

$$\tilde{\partial}(I_x) \subset \mathcal{I}_x + I. \tag{5}$$

Hence, $\mathcal{A}$ is L-graded. ☐

Example. Let $V_0 = V_a = \mathbb{Z}$ for all $a \in \mathbb{A}(L)$ and $V_x = 0$ otherwise, and consider the complex V with $\partial|_{V_a} = \mathrm{id}$. If t is a generator of V_0, the quotient of $\mathcal{A}(L, V)$ by the ideal generated by $t - 1$ is isomorphic to the Orlik-Solomon algebra of L.

Note that $\mathcal{I}_0 = 0$, so that $\mathcal{A}_0 = S_0$ and $\tilde{\partial}(\mathcal{A}_a)$, $a \in \mathbb{A}(L)$, is the ideal of S_0 generated by $\partial(V_a)$.

Proposition 5. *Suppose that V is an admissible L-graded complex. Then $\mathcal{A}(V)$ is also admissible (as a complex of vector spaces). In particular, if the maps $\partial_1|_{V_a}$ are all injective, $\mathcal{O}(\operatorname{Sym} V_0, ((\partial V_a))) \simeq \mathcal{A}(V)$ is exact.*

Proof. For any $x \in L$ we have $S(V)_{\leq x} = S(V_{\leq x})$ and therefore $\mathcal{A}(V)_{\leq x} = S(V_{\leq x})/\mathcal{I}_{\leq x}$. Consider the complex $V'_{\leq x}$ which coincides with $V_{\leq x}$ in degrees at least one and has $V'_{\leq x,0} = \partial(V_{\leq x,1})$. Also, let W_0 be a complement to $\partial(V_{\leq x,1})$ in V_0. Then it is easy to see that $\mathcal{A}(V_{\leq x}) \simeq \operatorname{Sym} W_0 \otimes S(V'_{\leq x})/\mathcal{I}_{\leq x}(V'_{\leq x})$ as differential graded algebras. Therefore, it is enough to show the existence of an admissible homotopy on $\mathcal{A}(V'_{\leq x}) = S(V'_{\leq x})/\mathcal{I}_{\leq x}(V'_{\leq x})$. Let d be an admissible homotopy for $V_{\leq x}$. From d we can construct a (super-)derivation $\tilde{d}$ of $S(V'_{\leq x})$. Consider the alternative grading on this algebra obtained by assigning every element of $V'_{\leq x}$ degree one. Then clearly $\tilde{\partial}$ and $\tilde{d}$ preserve this grading, and on the degree n part $S(V'_{\le x})^{(n)}$ of $S(V'_{\le x})$ we have $\tilde{d}\tilde{\partial} + \tilde{\partial}\tilde{d} = n\,\mathrm{id}$. It follows that by setting $d' = n^{-1}\tilde{d}$ on $S(V'_{\le x})^{(n)}$, $n \ge 1$, we obtain a contracting homotopy for $\tilde{\partial}$ (which is only defined on $\operatorname{im}\tilde{\partial}_1$ at the 0-th position). Observe that by (5) we have $\mathcal{I}_{\le x} = I_{\le x} + \tilde{\partial}(I_{\le x})$. Since d is admissible and whenever $y_1, \dots, y_k$ are dependent and $y'_1 \ge y_1$ then $y'_1, y_2, \dots, y_k$ are also dependent, we conclude that $\tilde{d}$ and d' preserve $\mathcal{I}_{\le x}$. It follows that d' defines an admissible homotopy for $\mathcal{A}(V'_{\le x})$. ☐

We are interested in the case where A is a hyperplane arrangement in a vector space U^* defined by linear functionals $u_a \in U$, L its intersection lattice and V the relation complex of A. If the complex V is admissible, by Proposition 5 the minimal complex associated to the atomic datum $(\operatorname{Sym} U, ((u_a)))$, where $\operatorname{Sym} U$ is the algebra of polynomials on U^*, and (u_a) the ideal of $\operatorname{Sym} U$ consisting of all polynomials vanishing on the hyperplane H_a, is exact.

Further cases can be derived by the application of a lattice-theoretic operation to admissible complexes.

Definition 7. Let L and Λ be geometric lattices, and $1 \le k \le n = \operatorname{rk} L$. We say that a monotone map $l : L_{\ge k} \to \Lambda$ is a *non-degenerate k-th truncation*, if

(1) $\operatorname{rk}_\Lambda(l(x)) = \operatorname{rk}_L(x) - k$ for all $x \in L_{\ge k}$,
(2) $l(x \vee_L y) = l(x) \vee_\Lambda l(y)$ for $x, y \in L_{\ge k}$ with $\operatorname{rk}_L(x \wedge y) \ge k$.

The most important example is the following. Let A be a hyperplane arrangement in a space U^* with intersection lattice L, and let $P \subset U$ be a subspace of codimension k. Assume that P satisfies the following non-degeneracy property:

$$x \cap P^\perp = \{0\}, \quad x \in L_k.$$

Then, denoting by π_P the projection $U^* \to P^*$, we have $\operatorname{codim} \pi_P(x) = \operatorname{rk} x - k$ for any $x \in L_{\ge k}$ and in particular the images under π_P of the elements of L_{k+1} (which are not necessarily distinct) form a new hyperplane arrangement A_P in the space P^*. If Λ is the intersection lattice of this arrangement, then the canonical map $l : L_{\ge k} \to \Lambda$, $x \mapsto \pi_P(x)$ is easily seen to be a non-degenerate k-th truncation map.

There is also a purely lattice-theoretic construction of truncations, the k-th Dilworth truncation [Bry86, Aig97]. It is a geometric lattice $L^{(k)}$ with a natural inclusion $L_{>k} \hookrightarrow L^{(k)}$ of posets which is a non-degenerate k-th truncation map. By [Bry85, Bry86], in the case of hyperplane arrangements the projection construction above yields in fact the k-th Dilworth truncation of L if the subspace P is generic, i.e. belongs to a Zariski open set of the corresponding Grassmannian.

The importance of truncation maps is explained by the following Lemma.

Lemma 2. *Let V be an admissible L-graded complex, and $l : L_{\ge k} \to \Lambda$ a non-degenerate k-th truncation map. Then the truncated shifted complex $V^{(k)} = (V_{i+k})_{i \ge 0}$ with the natural Λ-grading $V^{(k)}_\lambda := \oplus_{x \in L : l(x) = \lambda} V_x$, $\lambda \in \Lambda$, is an admissible Λ-graded complex.*

Proof. By our assumptions on l, the complex $V^{(k)}$ is compatibly Λ-graded. For any $\tilde{x} \in \Lambda$, the set $\{x \in L_{\ge k} : l(x) \le_\Lambda \tilde{x}\}$ is the union of L_k and the sets

$X_i = L_{\leq x_i} \cap L_{\geq k}$, $i = 1, \dots, m$, where $x_i \in L$ are the maximal elements with $l(x_i) \leq_\Lambda \tilde{x}$. By the second property of l, we have $\operatorname{rk}(x_i \wedge x_j) < k$ for $i \neq j$, and therefore the sets X_i are disjoint. This means that the obvious map of complexes

$$\oplus_{i=1}^m V_{\leq x_i}^{(k)} \longrightarrow V_{\leq \tilde{x}}^{(k)}$$

is an isomorphism except at the lowest point (corresponding to V_k), where we only have an inclusion of the left hand side into the right hand side. It follows that if d^i is an admissible homotopy for $V_{\leq x_i}$ then $d = \oplus_{i=1}^m d_i$ gives an admissible homotopy of $V_{\leq \tilde{x}}^{(k)}$. □

We can now apply these results to the case of hyperplane arrangements and relation spaces.

Corollary 2. *Let A be a hyperplane arrangement in the space U^* and V the relation complex of A. Let P be a subspace of U of codimension k such that $x \cap P^\perp = \{0\}$ for all $x \in L_k$. Let Λ be the intersection lattice of the hyperplane arrangement in P^* given by the hyperplanes $\pi_P(x) \subset P^*$, $x \in L_{k+1}$, and assume that these hyperplanes are all distinct. If V is admissible, then the minimal Λ-graded complex $\mathcal{O}(\operatorname{Sym} V_k, ((\partial V_{l^{-1}(a)}))_{a \in \mathbb{A}(\Lambda)})$ is exact.*

4. An exact sequence of Bernstein-Lunts type

We now turn to hyperplane arrangements over the field of real numbers. We quickly review their duality with zonotopes as explained in [Zie95, Ch. 7].

Let U be an n-dimensional real vector space and U^* its dual space, and A the hyperplane arrangement in U^* defined by linear functionals $u_a \in U$, $a \in \mathbb{A}$. Let H_a be the hyperplane of U^* defined by u_a. This arrangement induces a partition of U^* into the cones given by the connected components of $X \setminus \cup_{a: X \not\subset H_a}(X \cap H_a)$ where X ranges over the intersection lattice L of A. The set of all cones forms a lattice with $C_1 \leq C_2$ if and only if $\overline{C_1} \supset C_2$. There is a natural lattice map from the lattice of cones to the intersection lattice of A, which associates to each cone C the vector space spanned by it. Dually, we consider the zonotope (i.e. Minkowski sum of line segments) $\mathcal{Z} = \sum_a [-1, 1] u_a \subset U$, a convex polytope in the space U' spanned by the vectors u_a. If we map a cone C of A to the face

$$F = \{u \in \mathcal{Z} : \langle c, \cdot \rangle \text{ attains its maximum value on } \mathcal{Z} \text{ at } u\}$$

of $\mathcal{Z}$, where $c \in C$ is arbitrary, we obtain a lattice isomorphism between the lattice of cones of A and the face lattice of $\mathcal{Z}$, and under this duality the dimensions of C and F satisfy $\dim C + \dim F = n$. The induced lattice map from the faces of $\mathcal{Z}$ to the intersection lattice of A associates to F the space $x = x(F) = (F - F)^\perp \in L$, in which case we say that F is of type x. The zonotope $\mathcal{Z}$ is determined up to combinatorial equivalence (but not affine equivalence) by the hyperplane arrangement A.

Let $\mathfrak{E}_k$ be the set of k-dimensional faces of $\mathcal{Z}$, $k = -1, \dots, n$ (with $\mathfrak{E}_{-1} = \{\emptyset\}$ by convention). For any $x \in L$ the projection $\mathcal{Z}_{\geq x}$ of $\mathcal{Z}$ along $x^\perp$ is a zonotope dual to the restricted arrangement A^x. For $\operatorname{rk} x \leq i \leq n$, its faces of dimension

$i - \operatorname{rk} x$ are the projections of the i-dimensional faces of $\mathcal{Z}$ of type $\geq x$. We denote the set of the latter by $\mathfrak{E}_i^{\geq x}$.

Let now $\mathcal{D} = (M_0, (M_a))$ be an atomic datum over a ring R with respect to L. Using the 1-skeleton of $\mathcal{Z}$, which we can regard as a labeled graph with labels of the edges given by the atoms of L, we define a submodule of $M_0^{\mathfrak{E}_0}$ by congruence conditions along the edges:

$$\mathfrak{M} = \mathfrak{M}(\mathcal{Z}, \mathcal{D}) = \{m : \mathfrak{E}_0 \to M_0 : m(v) - m(v') \in M_{x(e)} \text{ for } v, v' \prec e \in \mathfrak{E}_1\}.$$

We are interested in the question whether a natural complex involving $\mathfrak{M}$ and graded by the face lattice of $\mathcal{Z}$ is exact. To construct this complex, fix an orientation on the vector space U. We can then consider the Euler-Poincare complex of $\mathcal{Z}$ with coefficients in M_0:

$$(EP_0) : 0 \to M_0^{\mathfrak{E}_{-1}} \xrightarrow{\delta_0} M_0^{\mathfrak{E}_0} \xrightarrow{\delta_1} M_0^{\mathfrak{E}_1} \xrightarrow{\delta_2} \dots \xrightarrow{\delta_n} M_0^{\mathfrak{E}_n} \to 0,$$

where the boundary maps are

$$\delta_j((m_{F'})_{F' \in \mathfrak{E}_{j-1}})_F = \sum_{F' \prec F} \operatorname{sign}(F', F) m_{F'}, \quad F \in \mathfrak{E}_j,$$

for $j = 0, \dots, n$, and the signs $\operatorname{sign}(F', F) \in \{\pm 1\}$ are determined by the fixed orientation. This complex is exact. Set $M_F = M^{x(F)}$ for any face F of $\mathcal{Z}$ and consider the natural subcomplex

$$(M_{\mathcal{D}}^{\mathcal{Z}}) \ : \ 0 \ \to \ M_0 \ \xrightarrow{\delta_0} \ \mathfrak{M} \ \xrightarrow{\delta_1} \ \oplus_{F \in \mathfrak{E}_1} M_F \ \xrightarrow{\delta_2} \ \oplus_{F \in \mathfrak{E}_2} M_F \ \xrightarrow{\delta_3} \ \dots \ \xrightarrow{\delta_n} \ M_{\mathcal{Z}} \ \to \ 0$$

of (EP_0). By the exactness of (EP_0) we have a short exact sequence

$$0 \to M_0 \to \mathfrak{M} \to \operatorname{Ker} \delta_2 \to 0. \tag{6}$$

In other words, $M_{\mathcal{D}}^{\mathcal{Z}}$ is exact at the first two places. By considering the cokernel of the natural inclusion $M_{\mathcal{D}}^{\mathcal{Z}} \hookrightarrow EP_0$, the exactness of $M_{\mathcal{D}}^{\mathcal{Z}}$ is equivalent to the exactness of

$$0 \to \mathfrak{M} \to \oplus_{F \in \mathfrak{E}_0} M_0 \xrightarrow{\delta_1} \oplus_{F \in \mathfrak{E}_1} M_0/M_F \xrightarrow{\delta_2} \oplus_{F \in \mathfrak{E}_2} M_0/M_F \xrightarrow{\delta_3} \dots \xrightarrow{\delta_n} M_0/M_{\mathcal{Z}} \to 0.$$

The following criterion allows us to reduce the question of exactness of $M_{\mathcal{D}}^{\mathcal{Z}}$ to the minimal complexes for the intersection lattice L studied earlier.

Proposition 6. *Let $N = \mathcal{O}(\mathcal{D}, L)$, and suppose that for all $x \in L$ the complex $(N_{\leq x}, \partial)$ is exact. Then the complex $M_{\mathcal{D}}^{\mathcal{Z}}$ is exact.*

Proof. For any i and $x \in L$ set

$$N_i^x = \begin{cases} \oplus_{y \leq x : \operatorname{rk}(y) = i} N_y, & i > 0, \\ M^x, & i = 0. \end{cases}$$

Also set $N_i^F = N_i^x$ for any face F of type x and $i = 0, \dots, n$. Clearly, $N_i^F \subset N_i^{F'}$ if $F \subset F'$. We will show by decreasing induction that for any $i = 0, \dots, n-1$ the complex

$$(D_i) : \oplus_{F \in \mathfrak{E}_{i+1}} \operatorname{Ker} \partial|_{N_i^F} \xrightarrow{\delta_{i+2}} \oplus_{F \in \mathfrak{E}_{i+2}} \operatorname{Ker} \partial|_{N_i^F} \to \dots \xrightarrow{\delta_n} \operatorname{Ker} \partial|_{N_i^{\mathcal{Z}}} \to 0$$

is exact. The statement is vacuous for $i = n-1$. To carry out the induction step, consider for any $x \in L_i$ the complex

$$(EP_x): \qquad \oplus_{F\in\mathfrak{E}_i^{\geq x}} N_x \to \oplus_{F\in\mathfrak{E}_{i+1}^{\geq x}} N_x \to \cdots \to N_x \to 0,$$

the Euler-Poincare complex of $\mathcal{Z}_{\geq x}$ with coefficients in N_x, which is exact. By our assumption, for any F of dimension $\geq i$ we have a short exact sequence

$$0 \to \operatorname{Ker} \partial_i|_{N_i^F} \to N_i^F \xrightarrow{\partial_i} \operatorname{Ker} \partial_{i-1}|_{N_{i-1}^F} \to 0.$$

Taking the direct sum over $F \in \mathfrak{E}_{i+j}$ we obtain the j-th column of a short exact sequence of complexes

$$0 \to D_i[1] \to \oplus_{x\in L_i} EP_x \xrightarrow{\partial} D_{i-1} \to 0.$$

The exactness of D_{i-1} follows from that of (EP_x), $x \in L_i$, by induction hypothesis.

For $i = 0$ we obtain the exactness of

$$\oplus_{e\in\mathcal{E}} M_e \to \oplus_{F\in\mathfrak{E}_2} M_F \to \cdots \to M_{\mathcal{Z}} \to 0.$$

Together with the exactness of (6) we obtain the Proposition. □

Remark 4. In the case $M_a = M_0$ for all $a \in \mathbb{A}(L)$ we can use this argument to prove the exactness of the Euler-Poincare complex with coefficients in M_0 by induction on the rank using the exactness of the Orlik-Solomon algebra.

Remark 5. Since the proof uses only the exactness of the Euler-Poincare complex, it carries over verbatim to general oriented matroids.

Remark 6. Let $\mathcal{D}$ be given by $M_0 = \operatorname{Sym} U$, $M_a = (u_a)$ for $a \in \mathbb{A}(L)$. Then $\mathfrak{M} = \mathfrak{M}(\mathcal{Z}, \mathcal{D})$ is the $\operatorname{Sym} U$-module (in fact algebra) of piecewise polynomial functions on A. Propositions 5 and 6 imply that the exact complex $M_{\mathcal{D}}^{\mathcal{Z}}$ is exact if the relation complex V is admissible, in particular for restrictions of Coxeter arrangements. By the work of Bernstein-Lunts [BL94, 15.7, 15.8] and Brion [Bri97, p. 12] it is known that $M_{\mathcal{D}}^{\mathcal{Z}}$ is exact for all simplicial hyperplane arrangements (which includes restrictions of Coxeter arrangements). The analogous statement remains in fact true for complete simplicial fans.

We now combine the results of the previous sections to show the main result of this paper. If M is a graded module over a polynomial ring R, we denote by $\operatorname{reg} M$ the Castelnuovo-Mumford regularity of M (cf. [Eis95]).

Theorem 1. *Let A be a hyperplane arrangement of rank n in a real vector space U^* with intersection lattice L, and $P \subset U$ a subspace of codimension k such that $x \cap P^{\perp} = \{0\}$ for all $x \in L_k$. Let $\pi_P : U^* \to P^*$ be the projection, and A_P be the hyperplane arrangement of rank $n-k$ in P^* given by the hyperplanes $\pi_P(x)$, $x \in L_{k+1}$, assumed to be all distinct. Let $\mathcal{Z}_P$ be a zonotope dual to A_P (of dimension $n-k$).*

Assume the relation complex V of A is admissible. Let $\mathcal{D}$ be the atomic datum over $R = \operatorname{Sym} V_k$ consisting of $M_0 = R$ and $M_a = (\partial V_x : \pi_P(x) = a)$ for $a \in \mathbb{A}(L_P)$, where L_P is the intersection lattice of A_P. Let $\mathfrak{M} = \mathfrak{M}(\mathcal{Z}_P, \mathcal{D})$. Then

(1) *The complex* $(M_{\mathcal{D}}^{\mathcal{Z}_P})$ *is exact.*
(2) $\operatorname{reg}\mathfrak{M} \le n-k$.

In particular, $\mathfrak{M}$ *is generated as a* $\operatorname{Sym} V_k$*-module by its homogeneous elements of degree* $\le n-k$.

Proof. The first part follows from Corollary 2 and Proposition 6.

For any face F of $\mathcal{Z}_P$ the ideal M_F of R is generated by linear functionals. Therefore $\operatorname{reg} M_F = 1$ for all F. The same is true for direct sums thereof. Since the complex $(M_{\mathcal{D}}^{\mathcal{Z}_P})$ is exact, we have short exact sequences

$$0 \to \operatorname{Ker}\delta_i \to \oplus_{F\in\mathfrak{C}_{i-1}} M_F \to \operatorname{Ker}\delta_{i+1} \to 0 \quad i=2,\dots,n-k.$$

(Recall that $\dim \mathcal{Z}_P = n-k$.) By the behavior of reg in short exact sequences (cf. [Eis95, Corollary 20.19]) we have

$$\operatorname{reg}(\mathfrak{M}) \le \operatorname{reg}(\operatorname{Ker}\delta_2),$$
$$\operatorname{reg}(\operatorname{Ker}\delta_i) \le \operatorname{reg}(\operatorname{Ker}\delta_{i+1}) + 1 \quad i=2,\dots,n-k.$$

Therefore $\operatorname{reg}(\mathfrak{M}) \le \operatorname{reg}(M_{\mathcal{Z}}) + n-k-1 = n-k$. □

Remark 7. If $\mathcal{Z}$ is a zonotope dual to A, a zonotope $\mathcal{Z}_P$ can be constructed directly as the fiber polytope in the sense of [BS92] of the projection of $\mathcal{Z}$ along P. This follows from considering $\mathcal{Z}$ as the projection of a hypercube of dimension $|A|$, and using [ibid., Lemma 2.3, Theorem 4.1]. The arrangement A_P and the zonotope $\mathcal{Z}_P$ govern the combinatorics of intersections of parallel translates of $P^\perp$ with the cone decomposition of U^* induced by A (cf. [BS94, Proposition 2.2]). If $k=1$, we obtain an arrangement governing the combinatorics of straight paths through A in a fixed direction. This is the case used in [FLM].

From Corollary 1 we infer:

Corollary 3. *The conclusion of the Theorem holds if* A *is a restriction of a real reflection arrangement.*

Question. Does the conclusion of the Theorem remain valid for simplicial hyperplane arrangements? It is not difficult to see, using the fact that simplicial arrangements are 2-formal [FR87], that this is at least the case for ranks $n \le 4$.

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