ON QUANTUM UNIQUE ERGODICITY FOR $\Gamma\backslash\mathbb{H}\times\mathbb{H}$

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1. Introduction

The purpose of this note is to point out a connection between the Quantum Unique Ergodicity (QUE) conjecture of Z. Rudnick and P. Sarnak [RS] (or, more precisely, natural higher rank generalizations of this conjecture) and conjectures of H. Furstenberg (unpublished), A. Katok and R. Spatzier [KS] and G. Margulis [M] regarding the scarcity of measures invariant under natural $\mathbb{R}^d$ actions ($d \geq 2$) in the ergodic theory of Lie groups, as well as improve and generalize on a related result of Rudnick and Sarnak (Theorem 1.6) .

Let M be a compact manifold, Δ the Laplacian on M.

Conjecture 1.1 (QUE [RS])**.** *If M is of negative curvature, then for any sequence of eigenfunctions ϕ_i of the Laplacian, normalized to have L^2-norm 1, such that the eigenvalues λ_i tend to $-\infty$, the probability measures $|\phi_i(x)|^2 d\,vol(x)$ converge in the weak* topology to the Riemannian volume $vol(M)^{-1} d\,vol$.*

(Recall that μ_i converge weak* to μ if for every continuous function with compact support,

$$\int f\,d\,\mu_i \longrightarrow \int f\,d\,\mu$$

as $i \to \infty$.) A similar conjecture can be stated also in the finite volume case [Sa].

A. Schnirelman [Sc], Y. Colin de Verdière [CdV], and S. Zelditch [Ze] showed for compact manifolds with ergodic geodesic flow (in particular for manifolds of negative curvature) that if one omits a subsequence of density zero of eigenvalues, the corresponding eigenfunctions obey the above conjecture. For the special case of $M = \Gamma\backslash\mathbb{H}$ with Γ a (congruence) arithmetic lattice in $PSL_2(\mathbb{R}) \subset Isom(\mathbb{H})$, and if the ϕ_i are assumed to be eigenfunctions of all Hecke operators associated with Γ, it follows from an identity of Watson [Wa] that the Grand Riemann Hypothesis implies that $|\phi_i(x)|^2 d\,vol(x)$ converge weak* to $vol(M)^{-1} d\,vol$. In fact, the GRH gives a best possible rate of convergence of these

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measures. ($\mathbb{H}$ denotes the hyperbolic upper half plane; for a definition of the Hecke operators see below.)

Consider now the analogous situation in the higher rank case. Let G be a semi-simple Lie group without compact factors and with finite center, Γ an irreducible lattice, and K a maximal compact subgroup of G. Take the symmetric space G/K as well as the finite volume quotient manifold $\Gamma\backslash G/K$. The Laplacian on G/K is G-invariant, but unless G is of rank 1, it does not generate the algebra of all G-invariant differential operators. In this paper we present some evidence to the following conjecture, for the special case $G = PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$.

Conjecture 1.2. *Let G, Γ, and K be as above. Let $\phi_i \in C^\infty(\Gamma\backslash G/K)$ be a sequence of eigenfunctions of the algebra of G-invariant differential operators, normalized to have L^2-norm 1, such that the eigenvalues of the Laplacian tend to $-\infty$. Then the probability measures $|\phi_i(x)|^2 d\,vol(x)$ converge weak* to the normalized Riemannian volume.*

This conjecture can be weakened by imposing additional conditions on the sequence ϕ_i. First we note that whenever Γ is arithmetic, and in particular in the higher rank case, there are additional natural G-invariant operators, the Hecke operators, commuting with the invariant differential operators, which can be defined as follows: Take a in the commensurator of Γ, set $L \subset G$ to be a set of representatives of the cosets of

$$\Gamma\backslash\Gamma a\Gamma,$$

and now define the Hecke operator T_a by

$$T_a\colon f(x) \mapsto \sum_{g\in L} f(gx).$$

A related more number theoretic definition of the Hecke operator can be found in §5.

- One can assume e.g. that the absolute value of the eigenvalues of the ϕ_i with respect to any G-invariant differential operator tends to ∞.
- As in the hyperbolic case one may assume that the ϕ_i are eigenfunctions of some or all of the Hecke operators.

We now restrict ourselves to the case of the rank 2 locally symmetric manifold $M = \Gamma\backslash\mathbb{H} \times \mathbb{H}$ with Γ an irreducible lattice in

$$PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R}) \subset Isom(\mathbb{H} \times \mathbb{H}),$$

although everything written here can be generalized to products of arbitrarily many upper half planes and probably without much difficulty to products of hyperbolic 2-spaces and 3-spaces. Here the algebra of invariant differentiable operators is generated by the partial Laplacian operators Δ_1 and Δ_2. By partial Laplacian operators we mean the differential operators on $\mathbb{H}\times\mathbb{H}$ defined by

$$\Delta_1(g\otimes f)=\Delta_{\mathbb{H}}(g)\otimes f \quad g,f\in C^\infty(\mathbb{H})$$

$$\Delta_1(g\otimes f)=g\otimes\Delta_{\mathbb{H}}(f),$$

$\Delta_{\mathbb{H}}$ being the hyperbolic Laplacian on $\mathbb{H}$. The Laplacian on $\Gamma\backslash\mathbb{H}\times\mathbb{H}$ is $\Delta_1+\Delta_2$. Furthermore, it is clear that Δ_1 and Δ_2 commute.

Let ϕ_i be a sequence of eigenfunctions of Δ_1 and Δ_2 with eigenvalues $\lambda_{1,i}$ and $\lambda_{2,i}$ respectively, such that the eigenvalues $\lambda_{1,i}+\lambda_{2,i}$ of the complete Laplacian tend to $-\infty$. Assume that $|\phi_i(x)|^2 d\,vol(x)$ tends to some probability measure μ' on M. The *micro local lift* is the basic construction employed by Schnirelman, Colin de Verdière, and Zelditch in the proof of their result mentioned earlier, and in this particular context will enable us to lift μ' to a probability measure μ on the unit cotangent bundle S^*M, or more precisely, allows us to construct from a subsequence of the ϕ_i a measure μ that projects to μ'. A basic property of the micro local lift is that the measure μ constructed in this way will be invariant under a 1-parameter flow — the geodesic flow.

However, in this case we can do better. As long as both $\lambda_{1,i}$ and $\lambda_{2,i}$ tend to $-\infty$, we can lift μ' to a measure μ on the 6-dimensional space $\Gamma\backslash PSL_2(\mathbb{R})\times PSL_2(\mathbb{R})$ (whereas S^*M is 7-dimensional). We define this lift in §3, where the main ingredient is a simplified version of Wolpert's elementary construction of the Zelditch micro local lift for quotients of $\mathbb{H}$. As we show in §4, the measure μ so obtained is invariant under the action from the right by the 2-parameter Cartan subgroup $A<PSL_2(\mathbb{R})\times PSL_2(\mathbb{R})$ where

$$A=\left\{\left(\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix},\begin{pmatrix}s&0\\0&s^{-1}\end{pmatrix}\right):t,s>0\right\}. \tag{1.1}$$

In particular, we prove the following:

Theorem 1.3. *Let ϕ_i be a sequence of eigenfunctions of Δ_1 and Δ_2 such that the eigenvalues $\lambda_{1,i}$ and $\lambda_{2,i}$ both tend to $-\infty$. Then any weak* limit of $|\phi_i(x)|^2 d\,vol(x)$ is the projection of an A invariant measure on $\Gamma\backslash PSL_2(\mathbb{R})\times PSL_2(\mathbb{R})$.*

It is reasonable to believe that this theorem should be true in the more general context considered in Conjecture 1.2.

While there are many measures on S^*M invariant under the geodesic flow, including many 'wild' ergodic measures which are supported on sets of fractal Hausdorff dimension, it is expected that the A invariant measures on $\Gamma\backslash PSL_2(\mathbb{R})\times PSL_2(\mathbb{R})$ are extremely rare – we quote the following conjecture of Margulis (as mentioned earlier in this introduction, there are similar conjectures by other authors):

Conjecture 1.4 ([M, Conjecture 2]). *Assume that G is a Lie group, Γ is a lattice in G, H is connected and generated by elements g such that all the eigenvalues of $Ad_G(g)$ are real. Let μ be an H-invariant H-ergodic measure on G/Γ. Suppose that for no $x \in \operatorname{supp} \mu$ there exists a closed connected subgroup $F < G$ and a continuous epimorphism of Lie groups $\psi : F \twoheadrightarrow L$ such that $H \subset F$, Fx is closed in $\Gamma\backslash G$, $\psi(F_x)$ is closed in L (where F_x denotes the stabilizer of x in F), and $\psi(H)$ is a one parameter subgroup of L containing no nontrivial Ad_L-unipotent elements.*

Then μ is algebraic, i.e. there is some closed connected $F < G$ such that Fx is closed in $\Gamma\backslash G$ and μ is Haar measure on $xF \cong (F \cap x^{-1}\Gamma x)\backslash F$.

For the special case of the action of the diagonal group A on $\Gamma\backslash PSL_2(\mathbb{R})\times PSL_2(\mathbb{R})$, Conjecture 1.4 implies for uniform Γ

Conjecture 1.5. *Any A ergodic invariant probability measure on $\Gamma\backslash PSL_2(\mathbb{R})\times PSL_2(\mathbb{R})$ is either Haar measure or Haar measure on a compact A orbit.*

We present a fairly standard argument deducing the above conjecture for uniform lattices from Conjecture 1.4 in the appendix. In addition to these conjectures, the results of [KS] give substantial restrictions on such measures unconditionally.

In section §5 we consider the question whether algebraic measures supported on compact A-orbit can appear in the decomposition of the A-invariant measure μ we obtain by lifting in Theorem 1.3 into its ergodic components, or what amounts to the same thing, whether the measure of a compact A-orbit (i.e. an invariant torus) can be strictly positive. Note that there are a coutable number of such invariant tori.

Several authors dealt with the analogous question in the rank 1 case; in this case the analogue of a compact A-orbit are the periodic orbits of the geodesic flow on $S^*(\Gamma\backslash\mathbb{H}) \cong \Gamma\backslash SL_2(\mathbb{R})$. Rudnick and Sarnak showed in the $PSL_2(\mathbb{R})$ case for certain uniform arithmetic lattices that if ϕ_i are eigenfunctions of the hyperbolic Laplacian on $\Gamma\backslash\mathbb{H}$ and also of all Hecke operators then the lift of any limiting measure is not supported on a finite set of compact orbits [RS]. Wolpert proved that if the singular support of a limiting measure of lifts of Hecke eigenforms on $PSL_2(\mathbb{Z})\backslash PSL_2(\mathbb{R})$ is compact, then the measure of any periodic orbit is 0 (indeed, under this assumption he proves a somewhat stronger statement) [Wo].

Here we show how to extend Rudnick and Sarnak's argument to show both for the geodesic flow in the rank 1 case and for the A-action on $\Gamma\backslash PSL_2(\mathbb{R})\times PSL_2(\mathbb{R})$ that the measure of a compact orbit according to a measure which is the lift of a limiting measure is 0. We do this for a fairly general class of lattices, namely uniform irreducible congruence lattices in $SL_2(\mathbb{R})^n$. In particular we obtain the following:

Theorem 1.6. *Let ϕ_j be a sequence of eigenfunctions on (say)*

- $\Gamma\backslash PSL_2(\mathbb{R})\times PSL_2(\mathbb{R})$ *of both Δ_1 and Δ_2,*
- $\Gamma\backslash PSL_2(\mathbb{R})$ *of the Laplacian*

with all eigenvalues tending in absolute value to infinity as $j\to\infty$, and assume further that ϕ_j are also eigenfunctions of all the Hecke operators, with Γ a uniform congruence lattice (see section §5 for more details). Let A be as in (1.1) *in the former case, and the subgroup of diagonal matricies of $PSL_2(\mathbb{R})$ in the latter. Then the measure of any compact A-orbit (i.e. an A-invariant torus or a periodic geodesic) according to the limiting measure μ obtained in Theorem 1.3 is 0.*

Corollary 1.7. *Conjecture 1.4 implies Conjecture 1.2 for $\Gamma\backslash\mathbb{H}\times\mathbb{H}$ and Γ a uniform lattice as above.*

Another unconditional corollary of Theorem 1.6 can be obtained by combining it with Theorem 1.1 from Lindenstrauss and Weiss [LW] (note that while stated for $SL_n(\mathbb{R})$, the proof of this result can be easily seen to work also $SL_2(\mathbb{R})\times SL_2(\mathbb{R})$, which is in fact simpler than the general case). For the special case of $\Gamma\backslash PSL_2(\mathbb{R})\times PSL_2(\mathbb{R})$, this theorem implies that the only A invariant closed subset of $\Gamma\backslash PSL_2(\mathbb{R})\times PSL_2(\mathbb{R})$ that contains a non isolated compact A orbit is $\Gamma\backslash PSL_2(\mathbb{R})\times PSL_2(\mathbb{R})$. Thus we get the following unconditional result:

Corollary 1.8. *Under the conditions of Theorem 1.6, if the support of μ contains a compact A-orbit then*

$$\operatorname{supp}\mu=\Gamma\backslash PSL_2(\mathbb{R})\times PSL_2(\mathbb{R}).$$

2. Notations

We let $G_1\cong G_2\cong PSL_2(\mathbb{R})$, $G=G_1\times G_2$. G_1 and G_2 can be considered as subsets of G in the obvious way. Take Γ to be an irreducible lattice in G. Let K_1, K_2, $K=K_1\times K_2$ be the usual maximal compact subgroups of G_1, G_2 and G respecitively. We will use the Iwasawa decomposition to represent every $g\in SL_2(\mathbb{R})\times SL_2(\mathbb{R})$ as

$$g=(g_1,g_2)$$

$$g_j = \begin{pmatrix} y_j^{1/2} & 0 \\ 0 & y_j^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & x_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix} \qquad j = 1, 2.$$

Of course, this also parametrizes G.

This representation, in particular, gives rise to a coordinate system on $\mathbb{H} \times \mathbb{H} = G/K$ by (y_1, x_1, y_2, x_2) which is identical with the usual coordinates on $\mathbb{H} \times \mathbb{H}$ that is $(x_1 + iy_1, x_2 + iy_2)$. Furthermore, this identification is compatible with the action of Γ on G and $\mathbb{H} \times \mathbb{H}$, so we can identify functions on $\Gamma \backslash \mathbb{H} \times \mathbb{H}$ with K-invariant functions on $\Gamma \backslash G$.

Let $k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and define

$$A_{n,m} = \{ f \in C^\infty(\Gamma\backslash G) : f(x_1 k(\theta_1), x_2 k(\theta_2)) = e^{2i(n\theta_1 + m\theta_2)} f(x_1, x_2) \}$$

(in particular $A_{0,0}$ is the space of K invariant functions). It is easy to see that the K-finite function, i.e. those functions in the linear span of the $A_{n,m}$ are dense in $C(\Gamma\backslash G)$.

Of particular importance are the raising and lowering operators

$$E_j^{\pm} = \pm 2iy_j e^{\pm 2i\theta_j} (\frac{\partial}{\partial x_j} \mp i \frac{\partial}{\partial y_i}) \mp ie^{\pm 2i\theta_j} \frac{\partial}{\partial \theta_j}$$

E_1^+, for example, takes functions in $A_{n,m}$ to $A_{n+1,m}$.

Other operators we will need are H_j which generate the Cartan flow.

$$H_j = -2y_j \sin 2\theta_j \frac{\partial}{\partial x_j} + 2y_j \cos 2\theta_j \frac{\partial}{\partial y_j} + \sin 2\theta_j \frac{\partial}{\partial \theta_j}$$

i.e.

$$\frac{d}{dt} f(g_1 \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, g_2)|_{t=0} = H_1 f.$$

We note that $2H_j = E_j^+ + E_j^-$. We will also use X_j, the generators of a two parameter unipotent subgroup of G.

$$X_j = y_j \cos 2\theta_j \frac{\partial}{\partial x_i} + y_j \sin 2\theta_i \frac{\partial}{\partial y_j} + \sin^2\theta_j \frac{\partial}{\partial \theta_j}.$$

Finally, we will use the operators W_j

$$W_j = \frac{\partial}{\partial \theta_j}.$$

A last remark is that $E_j^- E_j^+$, or perhaps more importantly the Casimir element of $sl_2(\mathbb{R})$,

$$\omega_j = E_j^- E_j^+ - W^2 - 2iW$$

coincide with the Laplacian

$$\Delta_j = 4y_j^2\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right)$$

on the K-invariant functions.

Let now ϕ_i be some sequence of normalized eigenvectors of Δ_1 and Δ_2 with eigenvalues $-(1+r_{1,i}^2)$ and $-(1+r_{2,i}^2)$, such that both $r_{1,i} \to \infty$ and $r_{2,i} \to \infty$ (possibly at very different rates). To eliminate excessive indexing we omit the index i in ϕ, r_1, etc.

For any such ϕ we let

$$\phi_{0,0}(g) = \phi(gK)$$

and define $\phi_{2n,2l}$ so that they satisfy the following

$$\begin{aligned}
(ir_1 + 1 + 2n)\phi_{2n+2,2l} &= E_1^+\phi_{2n,2l}\\
(ir_1 + 1 - 2n)\phi_{2n-2,2l} &= E_1^-\phi_{2n,2l}\\
(ir_2 + 1 + 2l)\phi_{2n,2l+2} &= E_2^+\phi_{2n,2l}\\
(ir_2 + 1 - 2l)\phi_{2n,2l-2} &= E_2^-\phi_{2n,2l}
\end{aligned}$$

3. Defintion of the Micro Local Lift

We will assign to every Δ_1, Δ_2-eigenfunction ϕ a functional defined on all K-finite functions as follows:

$$(*) \quad I_\phi(f) = \lim_{N\to\infty} \langle f \sum_{n,l=-N}^{N} \phi_{2n,2l}, \phi_{0,0}\rangle$$

where $\langle a, b\rangle = \int\limits_{\Gamma\backslash G} a(g)\overline{b}(g)dg$. Note that for K-finite f the limit in $(*)$ is purely formal since

$$\langle f \sum_{n,l=-N}^{N} \phi_{2n,2l}, \phi_{0,0}\rangle$$

is constant for any $N \geq N_0$ if

$$f \in \sum_{n,l=N_0}^{N_0} A_{n,l}$$

by orthogonality of the weight spaces $A_{n,l}$. Also note that

$$I_\phi(f) = \int\limits_{\Gamma\backslash G/K} f(x)|\phi(x)|^2 dx \qquad \text{for } f \in A_{0,0}.$$

Theorem 3.1. *Set* $\Phi_N := \frac{1}{2N+1} \sum_{l,m=-N}^{N} \phi_{2l,2m}$. *Then for K-finite f*

$$I_\phi(f) = \langle f\Phi_N, \Phi_N\rangle + O_f(N^{-1}) + O_f(N \max(r_1^{-1}, r_2^{-1}))$$

Proof. Suppose

$$f \in \sum_{l,m=-N_0}^{N_0} A_{l,m}.$$

By definition,

$$\langle f\Phi_N, \Phi_N\rangle = \frac{1}{(2N+1)^2} \sum_{k,t,l,m=-N}^{N} \langle f\phi_{2l,2m}, \phi_{2k,2t}\rangle.$$

Since

$$\langle f\phi_{2l,2m}, \phi_{2k,2t}\rangle \neq 0$$

only if $\max(|l-k|, |m-t|) \leq N_0$, we have that $\langle f\Phi_N, \Phi_N\rangle = O_f(1)$. Thus without loss of generality, $N \max(r_1^{-1}, r_2^{-1}) \leq 1$.

We now observe that for $-N \leq l, m, k, t \leq N$,

$$(**) \quad \langle f\phi_{2l,2m}, \phi_{2k,2t}\rangle = \langle f\phi_{2l-2,2m}, \phi_{2k+2,2t}\rangle + O_f(Nr_1^{-1})$$

Indeed

$$\begin{aligned}
&\langle f\phi_{2l,2m}, \phi_{2k,2t}\rangle = \\
&\qquad = \frac{1}{(ir_1 + 2l - 1)(-ir_1 - 2k - 1)} \langle fE_1^+\phi_{2l-2,2m}, E_1^-\phi_{2k+2,2t}\rangle \\
&\qquad = \langle f(\frac{E_1^+}{ir_1})\phi_{2l-2,2m}, (\frac{E_1^-}{ir_1})\phi_{2k+2,2t}\rangle + O(\| f \|_\infty N{r_1}^{-1}) \\
&\qquad = \langle \frac{E_1^+}{ir_1}(f\phi_{2l-2,2m}), \frac{E_1^-}{ir_1}\phi_{2k+2,2t}\rangle + O\left(\| E_1^+ f \|_\infty r_1^{-1}\right) \\
&\qquad\qquad + O\left(\| f \|_\infty Nr_1^{-1}\right) \\
&\qquad = \langle f\phi_{2l-2,2m}, \frac{E_1^+E_1^-}{r_1^2}\phi_{2k+2,2t}\rangle + O_f(Nr_1^{-1}).
\end{aligned}$$

But

$$E_1^+ E_1^- \phi_{2k+2,2t} = -[r_1^2 + O(N^2)]\phi_{2k+2,2t}$$

and (**) is established. Iterating the above calculation at most N_0 times gives

$$\langle f\phi_{2l,2m},\phi_{2k,2t}\rangle = \langle f\phi_{2l-2k,2m-2t},\phi_{00}\rangle + O_f(N\max(r_1^{-1},r_2^{-1}))$$

and so

$$\langle f\Phi_N,\Phi_N\rangle = \sum_{l,m}^{N}\frac{(N-|l|+1)(N-|m|+1)}{(2N+1)^2}\langle f\phi_{2l,2m},\phi_{0,0}\rangle + O_f(N\max(r_1^{-1},r_2^{-1})).$$

Since for $|l|,|m|\le N_0$,

$$\frac{(N-|l|+1)(N-|m|+1)}{(2N+1)^2} = 1 + O_f(N^{-1})$$

and using the K-finiteness of f, we see that

$$\langle f\Phi_N,\Phi_N\rangle = \sum_{l,m}\langle fE\phi_{2l,2m},\phi_{00}\rangle + O_f(N^{-1}) + O_f(N\max(r_1^{-1},r_2^{-1})).$$

□

If for a dense subset of $C(\Gamma\backslash G)$ consisting only of K-finite functions $\lim_{r_1,r_2\to\infty} I_\phi(f)$ exists, then this limit is given by

$$I(f) := \lim_{r_1,r_2\to\infty} I_\phi(f) = \lim_{N\to\infty}\ \lim_{r_1,r_2\to\infty}\langle f\Phi_N,\Phi_N\rangle.$$

Since if $f\ge 0$ $\quad\langle f\Phi_N,\Phi_N\rangle\ge 0$, we see that there is a measure μ s.t. $I(f)=\int f d\mu$. This measure is a probability measure since $I_\phi(1)=1$ for all elements of our sequence. Therefore, we have the following:

Corollary 3.2. 1. *There is a measure μ_ϕ on $\Gamma\backslash G$ (namely $|\Phi_N|^2 d\,vol_{\Gamma\backslash G}$ where N is $\min(r_1^{1/2},r_2^{1/2})$) so that*

$$|I_\phi(f) - \int f\,d\mu_\phi| = O_f(\max(r_1^{-1/2},r_2^{-1/2})).$$

2. *Any weak* limit of the $I_\phi(\cdot)$ is a probability measure on $\Gamma\backslash G$.*

In particular the above corollary allows us to lift a weak* limit of the $|\phi|^2 d\,vol$ to a measure μ on $\Gamma\backslash G$, which, as we will see in the next section, has nice invariance properties.

4. Invariance Under the Cartan Flow.

Let

$$A = \left\{ \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \right) : t, s > 0 \right\},$$

$$A_i = A \cap G_i \ for\ i = 1, 2.$$

Theorem 4.1. *Any μ obtained as in Corollary 3.2 is invariant under A.*

Theorem 4.1 is equivalent to having $\int (H_i f) d\mu = 0$ for $i = 1, 2, \quad f \in C^\infty(\Gamma\backslash G)$ since H_i is the generator for the corresponding one parameter flow obtained by multiplication on the right by elements of A_i. This, in turn, is equivalent to $I_\phi(H_i f) \to 0$ as $r_1, r_2 \to \infty$, but we have in fact the following exact statement, which for the 2 dimensional case was found by Zelditch. Since to prove Theorem 3 it is enough to consider f which are K-finite, we consider only such functions in the proposition below.

Proposition 4.2. *For $j = 1, 2$*

$$I_\phi(2ir_j H_j + H_j^2 + 4X_j^2)f) = 0;$$

In particular, $I_\phi(H_j f) = O_f(r_j^{-1})$

Proof. We assume through the proof that $j = 1$, and use $E^+, H, \dots$ to denote E_1^+, H_1, etc. The proof of this proposition is a simple calculation.

Write, formally, $\Phi_\infty = \sum_{n,m=-\infty}^{\infty} \phi_{2n,2m}$. This sum can be given an explicit meaning, but in fact in the following arguments can be replaced by a sufficiently big finite sum at the cost of additional indexation. The following identities are easily verified:

$$I_\phi(f) = \langle f\Phi_\infty, \phi_{0,0} \rangle$$

$$\langle (Wf)\Phi_\infty, \phi_{0,0} \rangle = -\langle f(W\Phi_\infty), \phi_{0,0} \rangle$$

$$E^+\Phi_\infty = (ir_1 - iW - 1)\Phi_\infty$$

$$E^-\Phi_\infty = (ir_1 + iW - 1)\Phi_\infty$$

$$E^-E^+\phi_\infty = (-r_1^2 - (iW - 1)^2)\Phi_\infty$$

We further recall that $E^- + E^+ = 2H$, where H is the generator of the 1-parameter Cartan action. Thus (with $2H = E^+ + E^-, 2iV = E^+ - E^-$, and $X = \frac{W+V}{2}$)

$$-(r_1^2+1)I_\phi(f) = \langle f\Phi_\infty, E^-E^+\phi_{0,0}\rangle = \langle E^-E^+f\Phi_\infty, \phi_{0,0}\rangle =$$

$$= \langle f(E^-E^+\Phi_\infty)+(E^+f)(E^-\Phi_\infty)+(E^-f)(E^+\Phi_\infty)+(E^-E^+f)\Phi_\infty, \phi_{0,0}\rangle =$$

$$= -(r_1^2+1)I_\phi(f) + 2ir_1 I_\phi(Hf)+$$

$$\left\langle \left[(W^2 - 2iW + E^-E^+ - 2H - 2WV)f\right]\Phi_\infty, \phi_{0,0}\right\rangle =$$

$$= -(r_1^2+1)I_\phi(f) + 2ir_1 I_\phi(Hf) + I_\phi((H^2+(W+V)^2)f)$$

and so

$$I_\phi(Hf) = \frac{i}{2r_1} I_\phi((H^2+4X^2)f)$$

□

5. MEASURES OF INVARIANT TORI

While it is possible to define and discuss the Hecke operators without invoking the more modern adelic language, using this language greatly simplifies the analysis of these operators, particularly when the class number of the quaternion algebra we are considering is greater than one. While this assumption is quite natural for $\Gamma\backslash\mathbb{H}$, it seems overly restrictive for $\Gamma\backslash\mathbb{H}\times\mathbb{H}$. To avoid having to deal with groups with class number greater than one, we shall deviate from [RS] by working with the group $SL_2(\mathbb{R})$ (and products of such) instead of $PGL_2(\mathbb{R})$. Our treatment here is quite general and will work for any irreducible congruence lattice in the product of any number of $SL_2(\mathbb{R})$ (the case of $SL_2(\mathbb{R})$ included).

We start by defining the manifolds on which we show no scarring occurs. We start from a quaternion algebra over a number field L, i.e. an algebra $H(L)$ whose elements are

$$\alpha = X_0 + X_1\omega + X_2\Omega + X_3\omega\Omega$$

where

$$\omega^2 = a; \Omega^2 = b, \omega\Omega = -\Omega\omega \tag{5.1}$$

with $a, b \epsilon \mathcal{O}_L \equiv \mathcal{O}$. The trace $tr(\alpha)$ and norm $n(\alpha)$ for $\alpha \in H(L)$ are defined in the usual way.

We require further that L is totally real (i.e. that all the embeddings $L \to \mathbb{C}$ take only real values), and let V_∞^r, V_∞^u and V_f denotes the infinite places ν of L where a H is ramified, i.e. $H(L_\nu)$ is a division ring, the infinite places at which H does not ramify, hence

$$H(L_\nu) \cong M_2(\mathbb{R}) \tag{5.2}$$

and the finite places of L respectively. Recall that $V_\infty^r \cup V_\infty^u$ can be identified with the embeddings $L \to \mathbb{R}$, and V_f with the prime ideas in L. Let G be the algebraic group over L of norm 1 quaternions in H. We now take $n = |V_\infty^u|$

$$\begin{aligned} G_\infty^u &= \prod_{\nu \in V_\infty^u} G(L_\nu) \\ G_\infty^r &= \prod_{\nu \in V_\infty^r} G(L_\nu) \\ G_f &= \prod_{\nu \in V_f} G(L_\nu) \\ X &= G(L) \backslash G(\mathbb{A}) / (G_\infty^r \times \tilde{G}) \end{aligned} \tag{5.3}$$

and $\tilde{G}$ to be a compact open subgroup of G_f.

Since G is simply connected (as an algebraic group), it follows from the Strong Approximization Theorem [RP, Chapter 7] that X is connected and is of the form

$$\Gamma \backslash G_\infty^u \cong \Gamma \backslash SL_2(\mathbb{R})^h$$

where

$$\Gamma = G(L) \cap [G_\infty^u \times G_\infty^r \times \tilde{G}]$$

where we identify $G(L)$ with both its image in $G(\mathbb{A})$ and G_∞^u, and similarly for $\Gamma < G(L)$.

We remark that for any $\gamma \in \Gamma < G(L)$ (considered as an element of $H(L)$),

$$tr(\gamma), n(\gamma) \in \mathcal{O}_L$$

We fix some isomorphism map $G_\infty^u \cong SL_2(\mathbb{R})^n$, and as in other sections of this note, take A to denote the positive diagonal subgroup in $SL_2(\mathbb{R})^n$, and take K to be the maximal compact

$$SO(2, \mathbb{R})^n \cong K < G_\infty^u.$$

We set $M = X/K$, a quotient manifold of $\mathbb{H}^n$. Similarly to the $n = 2$ case, the algebra of invariant differential operators on M is spanned by the Laplacian acting on each coordinate of $\mathbb{H}^n$, which we denote by $\Delta_1, \dots, \Delta_n$.

There is a standard way to attach to all but finitely many prime ideals $\nu \in V_f$ a Hecke operator; one way to do this is as follows: Equation (5.1) can be used to get a concrete model for G as a subvariety of GL_4; by the definition of the adelic topology, $\tilde{G}$ can be presented as

$$\widetilde{G} = \widetilde{G}_0 \times \prod_{\nu\in V_f\setminus S} G(\mathcal{O}_{L_\nu})$$

with $\tilde{G}_0$ an open compact subset of $\prod_{\nu\in S} G(L_\nu)$. We may further assume that H does not ramify for all $\nu \in V_f \setminus S$, and that we can identify $G(L_\nu)$ with $SL_2(L_\nu)$ and $G(\mathcal{O}_{L_\nu})$ with $SL_2(\mathcal{O}_{L_\nu})$. Let $t_\nu \in \mathcal{O}_\nu$ be a generator for $\nu\mathcal{O}_\nu$ in $\mathcal{O}_\nu$. Set $a_\nu \in G(\mathbb{A})$ by

$$[a_\nu]_{\nu'} = \begin{cases} e & \text{for } \nu' \neq \nu; \\ \begin{pmatrix} t_\nu & \\ & t_\nu^{-1} \end{pmatrix} & \text{for } \nu' = \nu, \end{cases}$$

and let π be the projection $G(K)^{S(\mathbb{A})} \to X$. For any $x \in X$, the set $\pi(\pi^{-1}(x)a_\nu^k)$ contains exactly $[\mathcal{O} : \nu]^{2k-1}([\mathcal{O} : \nu] - 1)$ points (and does not depend on the specific isomorphism $G(L_\nu) \cong SL_2(L_\nu)$ we use). We will denote these points by $T_{\nu^2 k}(x)$, and this also defines an operator on $L^2(X)$ by

$$T_{\nu^{2k}} : f(x) \mapsto \sum_{y\in T_{\nu 2k}(\gamma)} f(g)$$

For our purposes, we will need to give a slightly more explicit description of the Hecke points: take $\tilde{x} \in G^u_\infty$ so that $x = \Gamma\tilde{x}$. A point $y \in \Gamma_{\nu^2 k}(x)$ if and only if there is some $h \in G(L) \hookrightarrow \prod_{\nu'\in V_f} G(L_\nu)$

$$h \in \widetilde{G}a_\nu^k\widetilde{G} \cap G(L)$$

such that $y = \Gamma h\tilde{x}$. As an element in the quaternion algebra, it is clear that

$$n(h) = 1;$$

$$tr(h) \in \nu^{-k},$$

since if $tr(h') \notin \mathcal{O}_{L_\nu}$ the image of the sequence h'^n in $G(L_\nu)$ is unbounded, and $\tilde{G}$ is compact.

Note that an immediate consequence of this definition is that Hecke operators corresponding to different prime ideals commute. Another consequence is that the Hecke operators are self adjoint, or what amounts to the same

$$x \in T_{\nu^2}(y) \text{iff} y \in T_{\nu^2}(x)$$

A very similar construction allows us to define the Hecke operators also on $L^2(M)$ (which we will also denote in the same way).

Theorem 5.1. *Let M, X, be as above. Let ϕ_i be a sequence of joints eigenfunctions of $\Delta_1, \Delta_2, \dots, \Delta_n$ and all Hecke operators T_{ν^2}. Let μ be a weak* limit to the micro local lifts of the measures $|\phi_i|^2 d\,vol$. Then for any $\tilde{x} \in X$ whose A-orbit $\tilde{x}A$ is an invariant tori*

$$\mu(\tilde{x}A) = 0. \tag{5.4}$$

The proof of this theorem rests on the following lemma, as well as the well-known fact that there are no eigenfunctions of the Laplician on a regular tree with finite L^2-norm.

Lemma 5.2. *Let $\tilde{x} \in X$ be a point such that $\tilde{x}A$ is an invariant torus. Then there are infinitely many $\nu \in V_f$ for which for some open $e \in B \subset A$ the toral pieces*

$$\{\tilde{x}B, T_{\nu^2}(\tilde{x})B, T_{\nu^4}(\tilde{x})B, \dots\}$$

are all disjoint.

Proof. Assume that $\tilde{x}A$ is an invariant tori with $\tilde{x} = \Gamma x$. Then there is a $\gamma \in \Gamma$ such that

$$\gamma x \in xA \tag{5.5}$$

(in fact, the set of such γ is an n-dimensional commutative subgroup of $G(L)$). We can identify $\gamma \in \Gamma < G(L)$ with an element in the quaternion algebra $H(L)$. Then $L(\gamma) < H(L)$ is isomorphic to a number L', $[L' : L] = 2$, with γ mapped by the isomorphisms to $\alpha \in \mathcal{O}_{L'}$.

For every $\nu \in V_\infty^r$ the image of γ in $G(L_\nu)$ has (as an element in a compact group) only eigenvalues on the units circle. Note that the eigenvalues of this image are exactly the two possible images of α by embeddings τ of L' coinciding with ν on L, each appearing with multiplicity two; in particular τ is complex. On the other hand, by (5.5), for every $\nu \in V_\infty^u$ the eigenvalues of γ in $G(L_\nu)$ (which are exactly the eigenvalues of the 2-by-2 matrix γ is sent to by the isomorphisms $G(L_\nu) \cong SL_2(\mathbb{R})$ are in $\mathbb{R}$. Using the fact that the image of the natural embedding of L' into the product of all its infinite valuations is discrete, we see that there is a $\eta > 0$ such that for any $\alpha' \in \mathcal{O}_{L'}$, for at least one embedding τ of L' to $\mathbb{R}$ or $\mathbb{C}$

$$|\log|\tau(\alpha)|| > 2\eta.$$

Let B be defined by

$$B = B^{-1} = \left\{ \left(\begin{pmatrix} e^{s_1} & 0 \\ 0 & e^{-s_1} \end{pmatrix}, \dots, \begin{pmatrix} e^{s_n} & 0 \\ 0 & e^{-s_n} \end{pmatrix} \right) : \forall i \ |s_i| < \frac{\eta}{2} \right\}$$

From now to the end of the proof, we fix $\nu \in V_f \setminus S$, i.e. a prime ideal in L, that in addition *remains inert in* L'. It follows immediately from the Cebotarev Density Theorem (See [C], p. 159) that there are infinitely many such prime ideals in L (indeed, one can probably give an easier direct proof for this fact along the lines of the argument in [RS]).

Assume now that for some $\Gamma y \neq \Gamma y' \in \bigcup_{k=0}^{\infty} T_{\nu^{2k}}(\tilde{x})$,

$$yB \cap y'B \neq \emptyset. \tag{5.6}$$

Note that while L' and B depends on x, they remain fixed if we replace x with any other point in $G(L)x$. This is because $\Gamma \cap xAx^{-1}$ and $\Gamma \cap hxAx^{-1}h^{-1}$ are comeasurable for any $H \in G(L)$. Thus we may assume $x = y'$ and define k by $\Gamma y \in T_{\nu^{2k}}(\tilde{x})$, where as before $\tilde{x} = \Gamma x$.

From the explicit description of the Hecke points, we know that y' can be presented as hx with $h \in G(L) < H(L)$ satisfying $tr(h) \in \nu^{-k}$ and (like any element in $G(L)$) $n(h) = 1$. Since

$$hx \in xB^2 \subset xA$$

it follows from (5.5) and the fact that A is a commutative group that $[\gamma, h] = 0$. This implies that h is in the sub algebra $L[\gamma] \cong L'$ of $H(L)$ (otherwise $L(\gamma, h)$ would be an extension of L that properly contains L', and hence $H(L)$ would contain an element whose minimal polynomial over L is of degree ≥ 3 – which is of course impossible). From the minimal polynomial of h,

$$h^2 - tr(h)h + n(h) = 0$$

we know that $h \in \nu^{-k}\mathcal{O}_{L'}$ and that $n_{L'/L}(h) = 1$. Since ν is inert, $\nu\mathcal{O}_{L'} \lhd \mathcal{O}_{L'}$ is a prime ideal and $\bar{h} \in \nu^{-k}\mathcal{O}_{L'}$. The only way to reconcile these conditions is if $h \in \mathcal{O}_{L'}^*$. From the same reasoning we applied to γ, for any embedding $\tau : L' \to \mathbb{C}$ extending an embedding from V_∞^r, we have that $|\tau(h)| = 1$. Thus unless $h = 1$, which is clearly ridiculous, hx cannot be in xB^2, by definition of B. $\square$

Next we cite the following basic theorem about harmonic analysis on trees. Let $\mathcal{T}_d$ be a $d{+}1$-regular tree, and define $S_{\mathcal{T}_d} \colon L^2(\mathcal{T}_d) \to L^2(\mathcal{T}_d)$ by replacing the value of $f \in L^2(\mathcal{T}_d)$ at any node of the tree with the sum of the value of f on all of its neighbors. This operator is essentially the

Laplacian on $\mathcal{T}_d$, and is well-defined for any locally bounded function on $\mathcal{T}_d$. For more background on harmonic analysis on trees see [FTN].

Theorem 5.3. *$S_{\mathcal{T}_d}$ has no discrete spectrum (i.e. has no L^2-normalized eigenfunctions).*

The following corollary is an obvious consequence of the above theorem, using standard compactness arguments. While we do not need this, it can be seen that R can be taken to be $O(\epsilon^{-1})$ independently of the degree of the tree $\mathcal{T}_d$.

Corollary 5.4. *Fix $o \in \mathcal{T}_d$. For any $\epsilon > 0$ there is a R such that if $f \in \mathbb{R}^{\mathcal{T}}_d$ is an eigenfunction of $S_{\mathcal{T}_d}$ (note we do not assume $f \in L^2(\mathcal{T}_d)$),*

$$|f(0)|^2 \leq \epsilon \sum_{\substack{y \in \mathcal{T}_d, \\ 1 \leq d(x,y) \leq R}} |f(y)|^2$$

Proof of Theorem 5.1. The first thing we need to understand is how the assumption that ϕ_i are Hecke eigenfunctions is manifested in the properties of μ, the weak* limit of their micro local lifts.

Our starting point is Theorem 3.1. Note that purely from notational reasons this theorem is stated for joint eigenfunctions of the algebra of invariant differential operators only for the case of $n = 2$, i.e. $G^u_\infty \cong SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ – the proof works essentially as written also for the general case. In order not to introduce new and contradictory notations we shall assume that $n = 2$ also in this proof; but this assumption is in no way essential, and the proof carries out verbatim also for the general case, e.g. the much studied case of $n = 1$.

A key player in this theorem is the function $\Phi^{(j)}_N$ (the dependence on j in Theorem 3.1 is implicit)

$$\Phi^{(j)}_N = \frac{1}{2N+1} \sum_{a,b=-N}^{N} (\varphi_j)_{2a,2b}$$

for the general case, we sum over n indices instead of 2, and normalize accordingly. If the ϕ_j are Hecke eigenfunctions on M, then $\Phi^{(j)}_N$ are Hecke eigenfunctions on X; furthermore, by Theorem 3.1 for any $f \in C^\infty(X)$

$$\int f d\mu = \lim_{N\to\infty} \lim_{j\to\infty} \int f |\Phi^{(j)}_N|^2 dvol_X. \tag{5.7}$$

Suppose now that $\tilde{x}A = \Gamma x A$ is an invariant measure, and let $\nu \in V_f$ and $B \subset A$ be as in Lemma 5.2. Take $\epsilon > 0$ arbitrary, R depending on

ϵ as in Corollary 5.4. Take $B' \subset G^u_\infty$ to be an open set containing B such that

$$\{\tilde{y}B^1 : \tilde{y} \in Y_{x,R}\}$$

with

$$Y_{x,R} = \bigcup_{i=1}^{R} T_{\nu^{2i}}(\widetilde{x})$$

are all disjoint. For any $\tilde{y} \in X$, we can view

$$\{\tilde{y}\} \cup \bigcup_{i=1}^{\infty} T_{\nu^{2i}}(\tilde{y})$$

as a regular tree $\mathcal{T}_d$ of degree $d = [\nu : \mathcal{O}_L]([\nu : \mathcal{O}_L] + 1)$, whose edges are $(\tilde{y}', \tilde{y}'')$ with $\tilde{y}' \in T_{\nu^2}(\tilde{y}'')$. The Hecke operator T_{ν^2} is identical to $S_{\mathcal{T}_d}$ on every such Hecke-tree. Thus we see that for any N and $b \in B'$

$$|\Phi_N^{(j)}(\tilde{x}b)|^2 \leq \epsilon \sum_{\tilde{y} \in Y_{x,R}} |\Phi_N^{(j)}(\tilde{y}b)|^2$$

hence

$$\begin{aligned}\int_{\tilde{x}B^1} |\Phi_N^{(j)}|^2 dvol_X &\leq \epsilon \sum_{\tilde{y} \in Y_x, R} \int_{\tilde{y}B^1} |\Phi_N^{(j)}|^2 dvol_X \\ &\leq \epsilon \int_X |\Phi_N^{(j)}|^2 dvol_X = \epsilon\end{aligned}$$

and

$$\mu(\tilde{x}B) = 0.$$

□

Appendix A. Relation to the standard micro local lift

In this appendix we omit the proofs of the statements since they are not needed for any of the main results of this note.

Let ϕ_i be eigenfunctions of Δ_1 and Δ_2, with eigenvalues tending to $-\infty$, and let μ' be a weak* limit of $|\phi_i|^2 d\,vol$. We wish to relate the standard micro local lift μ'' of μ' to our lift μ. Recall that μ'' is a measure on S^*M (where M is $\Gamma\backslash\mathbb{H}\times\mathbb{H}$), invariant under the geodesic flow. On S^*M (equipped with the natural volume measure) the geodesic flow is not ergodic, with each ergodic component being the induced volume on a smooth submanifold of this space. Consider first $S^*(\mathbb{H}\times\mathbb{H})$, of which S^*M is a quotient. The points of this space are of the form (x_1, x_2, ξ_1, ξ_2), where $x_j \in \mathbb{H}$, and $\xi_j \in T^*_{x_j}(\mathbb{H})$ satisfying $\|\xi_1\|^2 + \|\xi_2\|^2 = 1$. The ratio $\|\xi_1\|/\|\xi_2\|$ is well defined on the quotient S^*M and is invariant under the geodesic flow. This gives us an invariant fibration of S^*M where all but two fibers are of codimension 1 and can be identified naturally with $\Gamma\backslash PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$. The exceptional fibers are identified with $\Gamma\backslash PSL_2(\mathbb{R}) \times \mathbb{H}$ and $\Gamma\backslash\mathbb{H} \times PSL_2(\mathbb{R})$. Under this identification the geodesic flow becomes the right action by a one-parameter subgroup of the Cartan group A (on each fiber a different one-parameter subgroup). It follows from the Howe-Moore Theorem [Zi] that with respect to the natural volume form on each fiber the geodesic flow is ergodic.

The micro local lift is defined whenever the ϕ_i are eigenfunctions of the Laplacian. Using the additional information that the ϕ_i are eigenfunctions of Δ_1 and Δ_2, one easily sees that μ'' is supported on one of the fibers in the above fibration of S^*M. The fiber on which μ'' is supported is determined by the limit ρ of $\lambda_{1,i}/\lambda_{2,i}$. The exceptional fibers correspond to $\rho = \infty$ and $\rho = 0$ respectively. Both of these facts follow easily by evaluating the integrals of $\|\xi_1\|^n$.

The identification of the regular fibers with $\Gamma\backslash PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ sends μ'' to our lift μ. However, from the standard construction, one can only get invariance under a one-parameter flow, whereas we get invariance under a two-parameter group.

If $\rho = 0$ the standard micro local lift will give us a measure on $\Gamma\backslash\mathbb{H} \times PSL_2(\mathbb{R})$; but as long as both $\lambda_{1,i}$ and $\lambda_{2,i}$ tend to $-\infty$ our lift gives us a measure on $\Gamma\backslash PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$. This measure is again invariant under a two-parameter group; its projection to $\Gamma\backslash\mathbb{H} \times PSL_2(\mathbb{R})$ can be identified with the standard micro local lift. The case $\rho = \infty$ is similar.

Appendix B. Conjecture 1.4 for $G = PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$

In this appendix we show that for uniform lattices Γ, Conjecture 1.4 does indeed imply that any A ergodic invariant measure on $\Gamma\backslash PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ is either Haar measure or Haar measure on a compact A orbit. We consider first the case Γ is co-compact. It is easy to classify the closed connected groups $A < F < G$ such that xF is closed. The Lie algebra of F is invariant under $Ad\,A$; therefore, it is a union of eigenspaces for $Ad\,A$. Let F' be the Lie group corresponding to the algebra generated by all eigenspaces on which $Ad\,A$ does not act trivially, or equivalently, F' is the group generated by all unipotent elements of F. One such eigenspace is

$$\left\{\left(\begin{pmatrix}0 & * \\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix}\right)\right\};$$

but if

$$\left\{\left(\begin{pmatrix}0 & * \\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix}\right)\right\} \in \mathit{Lie}\, F$$

then also

$$\left\{\left(\begin{pmatrix}0 & 0\\ * & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right)\right\}\in Lie\, F.$$

To see this note first that $\overline{xF'}\subset xF$ is by Ratner's theorem [R] homogeneous, i.e. there is a closed connected subgroup $H>F'$ so that $\overline{xF'}=xH$. Furthermore, xH is of finite volume (according to the induced metric on this submanifold).

Unless the above claim is true, the action of A does not preserves the volume of xH , and in fact since the dependence of $vol(exp(a)xH)$ on $a\in Lie\, A$ is linear, $vol(exp(a)xH)$ can be made to go to 0 which is a contradiction to the compactness of $\Gamma\backslash PSL_2(\mathbb{R})\times PSL_2(\mathbb{R})$. It only remains to eliminate the possibility that

$$F=\left\{\left(\begin{pmatrix}* & *\\ * & *\end{pmatrix},\begin{pmatrix}* & 0\\ 0 & *\end{pmatrix}\right)\right\}.$$

The unipotent radical of H is equal to the radical of H (see [Sh, §3] or the appendix in [LW]), and so $F'=H$. But this, in particular, this means that $|F'\cap x^{-1}\Gamma x|=\infty$, which shows that the projection of the irreducible lattice $x^{-1}\Gamma x$ from $PSL_2(\mathbb{R})\times PSL_2(\mathbb{R})$ to $PSL_2(\mathbb{R})$ is neither injective nor is the image finite, which is a contradiction to Margulis Normal Subgroup Theorem [Zi].

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