MEAN DIMENSION AND AN EMBEDDING PROBLEM: AN EXAMPLE

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Abstract. For any positive integer D, we construct a minimal dynamical system with mean dimension equal to $D/2$ that cannot be embedded into $(([0,1]^D)^{\mathbb{Z}}, \text{shift})$.

1. introduction

In this paper we study the problem of embedding a dynamical system (X, T) (a compact metric space X with a homeomorphism $T : X \to X$) into $(([0,1]^D)^{\mathbb{Z}}, \sigma)$. Here D is a positive integer, and $\sigma : ([0,1]^D)^{\mathbb{Z}} \to ([0,1]^D)^{\mathbb{Z}}$ is the shift transformation: $\sigma(x)_n = x_{n+1}$. "Embedding" means a topological embedding $f : X \to ([0,1]^D)^{\mathbb{Z}}$ satisfying $fT = \sigma f$.

An obvious obstacle for the embedding of a dynamical system (X, T) into $(([0,1]^D)^{\mathbb{Z}}, \sigma)$ is given by the set of periodic points of X: if the set $\mathrm{Peri}_n(X, T)$ of periodic points of period n cannot be topologically embedded into $[0,1]^{Dn}$ for some n, then (X, T) cannot be embedded into $(([0,1]^D)^{\mathbb{Z}}, \sigma)$ (an expanded discussion of this obstruction can be found in Gutman [4, Example 1.8].) When X has finite topological dimension Jaworski [6] (see also Auslander [1, Chapter 13, Theorem 9]) proved that if (X, T) has no periodic points then (X, T) can be embedded into the system $([0,1]^{\mathbb{Z}}, \sigma)$. Our main concern in this paper is the case of (X, T) minimal.

Mean dimension (denoted $\mathrm{mdim}(X, T)$) is a natural invariant of dynamical systems introduced by Gromov [3]. It is zero for finite dimensional systems, and is equal to D for the dynamical system $(([0,1]^D)^{\mathbb{Z}}, \sigma)$. Weiss and the first named author observed in [8] that mean dimension gives another, less obvious obstacle for embedding a dynamical system in another: namely, if (X, T) can be embedded into the system (Y, S) then $\mathrm{mdim}(X, T) \leq \mathrm{mdim}(Y, S)$. In particular, if (X, T) can be embedded in $(([0,1]^D)^{\mathbb{Z}}, \sigma)$ then $\mathrm{mdim}(X, T) \leq D$. A construction of a minimal dynamical system (which in particular has no periodic points) whose mean dimension is greater than 1 is given in [8, Proposition 3.3]; it follows that this system cannot be embedded into $([0,1]^{\mathbb{Z}}, \sigma)$ despite the fact that it has no periodic points.

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In [7] the first named author proved a partial converse to above necessary criterion for embeddedability of a dynamical system in $(([0,1]^D)^{\mathbb{Z}}, \sigma)$:

Theorem 1.1. *There exists a positive number $c \geq 1/36$ satisfying the following: If a dynamical system (X,T) is an extension of an infinite minimal system and* $\mathrm{mdim}(X,T) < cD$*, then (X,T) can be embedded into the system $(([0,1]^D)^{\mathbb{Z}}, \sigma)$.*

This raises the interesting problems of determining the optimal value of the positive constant c in the above statement.

Recall the following classical result in dimension theory ([5, p. 56, Theorem V 2]): If X is a compact metric space with $\dim X \leq D$, then X can be topologically embedded into $[0,1]^{2D+1}$. This motivates the following conjecture:

Conjecture 1.2. *Let (X,T) be a dynamical system so that for every n we have that* $\frac{1}{n}\dim(\mathrm{Peri}_n(X,T)) < D/2$ *and* $\mathrm{mdim}(X,T) < D/2$*. Then (X,T) can be embedded into the system $(([0,1]^D)^{\mathbb{Z}}, \sigma)$.*

The main result of this paper is the following.

Theorem 1.3. *Let D be a positive integer. There exists a minimal system (X,T) with* $\mathrm{mdim}(X,T) = D/2$ *but (X,T) cannot be embedded into the system $(([0,1]^D)^{\mathbb{Z}}, \sigma)$.*

This theorem shows that if Conjecture 1.2 is true then the condition $\mathrm{mdim}(X,T) < D/2$ is optimal.

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2. Some preliminaries

2.1. **Review of mean dimension.** We review the basic definitions of mean dimension; cf. Gromov [3] and Lindenstrauss-Weiss [8] for more details. Let (X,d) be a compact metric space. Let Y be a topological space, and let $f: X \to Y$ be a continuous map. For a positive number ε, we call f an ε-embedding if we have $\mathrm{Diam} f^{-1}(y) \leq \varepsilon$ for all $y \in Y$. We define $\mathrm{Widim}_\varepsilon(X,d)$ as the minimum integer $n \geq 0$ such that there exist an n-dimensional polyhedron (a topological space admitting a structure of simplicial complex) P and an ε-embedding $f: X \to P$. The following example (cf. Gromov [3, p. 332]) will be used later; for a proof, see [8, Lemma 3.2]:

Example 2.1.

$$\mathrm{Widim}_\varepsilon([0,1]^N, d_{\ell^\infty}) = N, \quad (0 < \varepsilon < 1),$$

where d_{ℓ^∞} is the ℓ^∞-distance: $d_{\ell^\infty}(x,y) = \max_i |x_i - y_i|$.

We also note the following lemma:

Lemma 2.2. *Let (X, d) and (Y, d') be compact metric spaces. Suppose that there is a continuous distance-increasing map from X to Y. Then* $\mathrm{Widim}_\varepsilon(X, d) \leq \mathrm{Widim}_\varepsilon(Y, d')$ *for all $\varepsilon > 0$. (A map $f : X \to Y$ is distance-increasing if $d(x, y) \leq d(f(x), f(y))$ for all $x, y \in X$.)*

Proof. If $f : X \to Y$ is distance-increasing and $g : Y \to P$ is an ε-embedding, then $g \circ f : X \to P$ is also an ε-embedding. □

Let $T : X \to X$ be a homeomorphism. For an integer $n \geq 0$, we define a distance d_n on X by $d_n(x, y) = \max_{0 \leq i < n} d(T^i x, T^i y)$. We define the mean dimension $\mathrm{mdim}(X, T)$ by

$$\mathrm{mdim}(X, T) := \lim_{\varepsilon \to 0} \left(\lim_{n \to \infty} \frac{\mathrm{Widim}_\varepsilon(X, d_n)}{n} \right).$$

The function $n \mapsto \mathrm{Widim}_\varepsilon(X, d_n)$ is subadditive. Hence the above limit exists. The mean dimension $\mathrm{mdim}(X, T)$ is a topological invariant, i.e., it is independent of the choice of a distance d compatible with the topology. The fundamental example is the following (for the proof, see [8, Proposition 3.3]):

Example 2.3. Let D be a positive integer. Consider the D-dimensional unit cube $[0, 1]^D$. Let $([0, 1]^D)^{\mathbb{Z}}$ be the infinite product of the copies of $[0, 1]^D$ indexed by $\mathbb{Z}$ with the product topology. Let $\sigma : ([0, 1]^D)^{\mathbb{Z}} \to ([0, 1]^D)^{\mathbb{Z}}$ be the shift transformation: $\sigma(x)_n = x_{n+1}$. Then

$$\mathrm{mdim}(([0, 1]^D)^{\mathbb{Z}}, \sigma) = D.$$

In section 3, we use a "block-type" system. This type of construction was used in the context of mean dimension by Weiss and the first named author in [8, Proposition 3.5] and by Coornaert-Krieger [2]. Let K be a compact metric space, and let b be a positive integer. Let $B \subset K^b$ be a closed subset. We define a block-type system $X(B) \subset K^{\mathbb{Z}}$ by

$$X(B) := \{x \in K^{\mathbb{Z}} | \, \exists k \in \mathbb{Z} \, \forall l \in \mathbb{Z} : x_{k+lb}^{k+lb+b-1} \in B\}$$

where $x_m^n = (x_m, x_{m+1}, \ldots, x_n)$ for $m \leq n$. Let $\sigma : X(B) \to X(B)$ be the shift transformation.

Lemma 2.4.
$$\mathrm{mdim}(X(B), \sigma) \leq \frac{\dim B}{b}.$$

Proof. Let d be a distance on K. We define a distance d' on $K^{\mathbb{Z}}$, compatible with the product topology, by

$$d'(x, y) := \sum_{n \in \mathbb{Z}} 2^{-|n|} d(x_n, y_n).$$

For $\varepsilon > 0$, we take a positive integer $L = L(\varepsilon)$ satisfying $\sum_{|n| > L} 2^{-|n|} < \varepsilon / \mathrm{Diam}(K)$. For a positive integer n, let $\pi_n : K^{\mathbb{Z}} \to K^{\{-L, -L+1, \ldots, n+L\}}$ be the natural projection. If two points $x, y \in K^{\mathbb{Z}}$ satisfy $\pi_n(x) = \pi_n(y)$, then $d'_n(x, y) < \varepsilon$.

We decompose $X(B)$: $X(B) = X_0 \cup X_1 \cup \cdots \cup X_{b-1}$ where

$$X_k = \{x \in K^{\mathbb{Z}} | \, \forall l \in \mathbb{Z} : x_{k+lb}^{k+lb+b-1} \in B\}.$$

By definition,

$$\dim \pi_n(X_k) \leq \frac{n \dim B}{b} + \mathit{const}$$

where *const* is a positive constant independent of n. Recall the Sum Theorem in dimension theory [5, p. 30]: if a compact metric space Y is a countable or finite union of closed sets Y_i then $\dim Y = \max_i \dim Y_i$. Applying this theorem to the sets X_i we obtain

$$\dim \pi_n(X(B)) \leq \frac{n \dim B}{b} + \mathit{const}.$$

Since $\pi_n : X(B) \to \pi_n(X(B))$ is an ε-embedding,

$$\mathrm{Widim}_\varepsilon(X(B), d'_n) \leq \dim \pi_n(X(B)) \leq \frac{n \dim B}{b} + \mathit{const}.$$

Hence

$$\mathrm{mdim}(X(B), \sigma) = \lim_{\varepsilon \to 0} \left(\lim_{n \to \infty} \frac{\mathrm{Widim}_\varepsilon(X(B), d'_n)}{n} \right) \leq \frac{\dim B}{b}.$$

□

2.2. **Topological preliminaries.** First we fix some notations. For a topological space X we define its **cone** $C(X)$ by $C(X) := [0,1] \times X/\sim$ where $(0,x) \sim (0,y)$ for all $x, y \in X$. The equivalence class of (t,x) is denoted by $[tx]$. We set σ_{n-1} to be the $(n-1)$-dimensional simplex

$$\sigma_{n-1} := \{(t_1, \ldots, t_n) \in \mathbb{R}^n | \, t_1, \ldots, t_n \geq 0, t_1 + \cdots + t_n = 1\}.$$

For topological spaces $X_1, \ldots, X_n$ we define their **join** $X_1 * \cdots * X_n$ by

$$X_1 * \cdots * X_n := \sigma_{n-1} \times X_1 \times \cdots \times X_n / \sim$$

where $(t_1, \ldots, t_n, x_1, \ldots, x_n) \sim (s_1, \ldots, s_n, y_1, \ldots, y_n)$ iff

$$t_i = s_i \ (\forall 1 \leq i \leq n) \quad \text{and} \quad x_i = y_i \ (\forall 1 \leq i \leq n \text{ satisfying } t_i \neq 0).$$

The equivalence class of $(t_1, \ldots, t_n, x_1, \ldots, x_n)$ is denoted by $t_1x_1 \oplus \cdots \oplus t_nx_n$. If $X_1, \ldots, X_n$ admit the structure of a simplicial complex, so does $X_1 * \cdots * X_n$ in a canonical way.

Let Y be the **triod graph**, i.e. the graph of the shape "Y". (Rigorous definition is as follows. Let D_3 be the 3-points discrete space, and set $Y := C(D_3)$: the cone of D_3.) Let d be the graph distance on Y (all three edges have length one). Let n be a positive integer, and let d_{ℓ^∞} be the ℓ^∞-distance on Y^n: $d_{\ell^\infty}(x,y) := \max_i d(x_i, y_i)$. It is known that Y^n cannot be topologically embedded into $\mathbb{R}^{2n-1}$. A proof of this result based on the Borsuk-Ulam theorem can be found in A.B. Skopenkov [10, pp. 287-288], and more general results on the problem of embedding products of graphs into the Euclidean spaces in M. Skopenkov [11]. The purpose of this subsection is to prove an ε-embedding version of the above result:

Proposition 2.5. *For any $0 < \varepsilon < 1$, there does not exist an ε-embedding from (Y^n, d_{ℓ^∞}) to $\mathbb{R}^{2n-1}$.*

It is likely this proposition is known to some specialists. The proof below is an application of the method in Matoušek's book [9, Chapter 5]; probably it also follows from the method of [10].

The most important ingredient of the proof is the following form of the Borsuk-Ulam theorem [9, p. 23, (BU2a)]: There does not exist a $\mathbb{Z}_2$-equivariant continuous map from S^n to S^{n-1}. Here $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, and it acts on S^n by the antipodal map.

Let K be a (geometric) simplicial complex; in this subsection we consider only finite simplicial complexes. For a point $x \in K$ we denote by $\mathrm{supp}(x)$ the simplex of K containing x in its relative interior. Let $K^{*2} := K * K$ be the join of the two copies of K, and define the **deleted join** $K^{*2}_\Delta \subset K^{*2}$ by

$$K^{*2}_\Delta := \{(1-t)x_1 \oplus tx_2 |\, 0 \le t \le 1, x_1, x_2 \in K, \mathrm{supp}(x_1) \cap \mathrm{supp}(x_2) = \emptyset\}.$$

By convention, for any $x \in K$ both $x \oplus 0$ and $0 \oplus x$ are contained in K^{*2}_Δ. The group $\mathbb{Z}_2$ freely acts on this space by $(1-t)x_1 \oplus tx_2 \mapsto tx_2 \oplus (1-t)x_1$. For example, $(D_3)^{*2}_\Delta$ is $\mathbb{Z}_2$-homeomorphic to S^1.

The following fact is easy to prove (see [9, 5.5.2 Lemma]): Let K and L be simplicial complexes. Then we have a $\mathbb{Z}_2$-homeomorphism:

$$(K * L)^{*2}_\Delta \cong K^{*2}_\Delta * L^{*2}_\Delta. \tag{1}$$

(In the right-hand-side, $\mathbb{Z}_2$ acts on K^{*2}_Δ and L^{*2}_Δ simultaneously.) The following lemma is proved in [9, 5.5.4 Lemma].

Lemma 2.6. *Set*

$$R_n := (\mathbb{R}^n)^{*2} \setminus \left\{\frac{1}{2}x \oplus \frac{1}{2}x |\, x \in \mathbb{R}^n\right\}.$$

$\mathbb{Z}_2$ acts on R_n as in the above. Then there is a $\mathbb{Z}_2$-equivariant continuous map from R_n to S^n.

Following Skopenkov [11, p. 193], we shall make use of the following fact:

Lemma 2.7. *Let K and L be simplicial complexes, and CK and CL be their cones. Then there is a homeomorphism $f : CK \times CL \to C(K * L)$ such that for any simplex $\Delta \subset C(K * L)$ its preimage $f^{-1}(\Delta)$ is a union of (at most two) sets of the form $\Delta_1 \times \Delta_2$ (Δ_1, Δ_2 are simplexes of CK, CL respectively). Here we use the natural simplicial complex structures of CK, CL and $C(K * L)$.*

Proof. Take a homeomorphism

$$\varphi : [0,1] \times [0,1] \to \{(S,T) |\, S \ge 0, T \ge 0, S + T \le 1\}, \quad (s,t) \mapsto (S,T),$$

such that $\varphi(\{s=0\}) = \{S=0\}$, $\varphi(\{t=0\}) = \{T=0\}$ and $\varphi(\{s=1\}\cup\{t=1\}) = \{S+T=1\}$. Let p, p', p'' be the base point of the cones $C(K*L)$, $C(K)$, and $C(L)$, so that we may identify $C(K*L)$ with $\{p\}*K*L$ (and similarly for p', p'' and $C(K), C(L)$). We define $f: CK\times CL \to C(K*L) = \{p\}*K*L$ by

$$f([sx],[ty]) := (1-S-T)p\oplus Sx\oplus Ty, \quad (0\leq s,t\leq 1, \varphi(s,t) = (S,T), x\in K, y\in L).$$

From $\varphi(\{s=0\}) = \{S=0\}$ and $\varphi(\{t=0\}) = \{T=0\}$, the map f is well-defined. Since φ is a homeomorphism, so is f. If Δ is a simplex in $C(K)$ (which we identify with the appropriate simplex in $C(K*L)$), then $f^{-1}(\Delta) = \Delta\times\{p''\}$ and similarly for $\Delta\subset C(L)$, $f^{-1}(\Delta) = \{p'\}\times\Delta$. Otherwise, $\Delta\subset C(K*L)$ may either contain p in which case it has the form $C(\Delta_1 * \Delta_2)$ or it does not in which case it has the form $\Delta_1*\Delta_2$ with $\Delta_1\subset K$ and $\Delta_2\subset L$. In the first case, $f^{-1}(C(\Delta_1*\Delta_2)) = C(\Delta_1)\times C(\Delta_2)$ whereas in the second

$$f^{-1}(\Delta_1*\Delta_2) = C(\Delta_1)\times\Delta_2\cup\Delta_1\times C(\Delta_2).$$

□

Proof of Proposition 2.5. $Y = C(D_3)$. By iterated applications of Lemma 2.7, there is a homeomorphism $f: Y^n\to C((D_3)^{*n}) =: K$ such that for any simplex $\Delta\subset K$ its preimage $f^{-1}(\Delta)$ is a union of sets of the form $\Delta_1\times\cdots\times\Delta_n$ ($\Delta_1,\dots,\Delta_n$ are simplexes of Y). This property implies: if $x, y\in K$ satisfy $\mathrm{supp}(x)\cap\mathrm{supp}(y) = \emptyset$ then $d_{\ell^\infty}(f^{-1}(x), f^{-1}(y))\geq 1$. (Note that the distance between two disjoint simplexes of Y is greater than or equal to 1.)

Suppose that there is an ε-embedding $\varphi: (Y^n, d_{\ell^\infty})\to\mathbb{R}^{2n-1}$ for $0<\varepsilon<1$. Then for any two points $x, y\in K$ with $\mathrm{supp}(x)\cap\mathrm{supp}(y)=\emptyset$ we have $\varphi\circ f^{-1}(x)\neq\varphi\circ f^{-1}(y)$. Then we can define a $\mathbb{Z}_2$-equivariant continuous map from the deleted join K^{*2}_Δ to R_{2n-1} (defined in Lemma 2.6) by $(1-t)x\oplus ty\mapsto(1-t)\varphi\circ f^{-1}(x)\oplus t\varphi\circ f^{-1}(y)$. Hence by Lemma 2.6 there is a $\mathbb{Z}_2$-equivariant continuous map from K^{*2}_Δ to S^{2n-1}. On the other hand, we have the following $\mathbb{Z}_2$-homeomorphisms:

$$K^{*2}_\Delta = (\{p\}*(D_3)^{*n})^{*2}_\Delta\cong(\{p\})^{*2}_\Delta*\{(D_3)^{*2}_\Delta\}^{*n}\cong S^0*(S^1)^{*n}\cong S^{2n}.$$

Here we have used the identification of $K = C((D_3)^{*n})$ with the join of $(D_3)^{*n}$ with a one-point space $\{p\}$, the identity (1), as well as the $\mathbb{Z}_2$-homeomorphisms $(\{p\})^{*2}_\Delta\cong S^0$, $(D_3)^{*2}_\Delta\cong S^1$ and $S^l*S^m\cong S^{l+m+1}$. Therefore we conclude that there is a $\mathbb{Z}_2$-equivariant continuous map from S^{2n} to S^{2n-1}. But this contradicts the Borsuk-Ulam theorem. □

3. Proof of the main theorem

The construction of X below is based on Lindenstrauss-Weiss [8, pp. 10-11]. Let Y be the triod graph. Let D be a positive integer, and set $K := Y^D$. Let d be the graph distance on Y, and let d_{ℓ^∞} be the ℓ^∞-distance on $K = Y^D$ introduced in Section 2.2. We define a distance on $K^{\mathbb{Z}}$ by

$$\mathrm{dist}(x,y) := \sum_{n\in\mathbb{Z}} 2^{-|n|} d_{\ell^\infty}(x_n, y_n).$$

Let $\sigma : K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ be the shift transformation.

Fix a sequence of rational numbers p_n $(n \geq 1)$ such that

$$\prod_{n=1}^{\infty}(1-p_n) = \frac{1}{2}, \quad 0 < p_n < 1. \tag{2}$$

We will construct the following three objects satisfying the conditions (i)-(iv) below:

- A decreasing sequence of closed shift-invariant subsets of $K^{\mathbb{Z}}$:

$$K^{\mathbb{Z}} = X_0 \supset X_1 \supset X_2 \supset X_3 \supset \dots, \quad X := \bigcap_{n=0}^{\infty} X_n.$$

- A sequence of integers:

$$0 = a_0 < b_0 < a_1 < b_1 < \dots < a_n < b_n < a_{n+1} < b_{n+1} < \dots.$$

- Closed subsets $B_n \subset K^{b_n}$ $(n \geq 1)$.

(i) $a_0 = 0$ and $b_0 = 1$. $p_n = a_n/b_n$ $(n \geq 1)$. $b_n \mid b_{n+1}$ and $b_n \mid a_{n+1}$. Moreover

$$b_n\left(2\prod_{k\leq n}(1-p_k) - 1\right) \to \infty \qquad \text{as } n \to \infty. \tag{3}$$

(ii) X_n is the block-type space defined by B_n:

$$X_n = \{x \in K^{\mathbb{Z}} |\, \exists k \in \mathbb{Z}\, \forall l \in \mathbb{Z} : x_{k+lb_n}^{k+lb_n+b_n-1} \in B_n\}.$$

$B_0 = K$ and $X_0 = K^{\mathbb{Z}}$.

(iii) We define a decreasing sequence $\mathbb{Z} = I_0 \supset I_1 \supset I_2 \supset \dots$ by

$$I_n := \{x \in \mathbb{Z} |\, \forall 0 \leq k \leq n,\ \exists j \in \{0, 1, \dots, b_k - a_k - 1\} : x \equiv j \bmod b_k\}.$$

We identify K^{b_n} with $K^{\{0,1,2,\dots,b_n-1\}}$, and let $\pi_n : K^{b_n} \to K^{\{0,1,2,\dots,b_n-1\}\setminus I_n}$ be the projection. Then there is $x(n) \in K^{\{0,1,2,\dots,b_n-1\}\setminus I_n}$ such that $B_n = \pi_n^{-1}(x(n))$ $(n \geq 1)$. The sequence $\{x(n)\}_{n\geq 1}$ satisfies the following compatibility condition: If $k \in \{0,1,2,\dots,b_{n+1}-1\}\setminus I_{n+1}$ and $k' \in \{0,1,2,\dots,b_n-1\} \setminus I_n$ satisfy $k \equiv k' \bmod b_n$, then $x(n+1)_k = x(n)_{k'}$. This condition is equivalent to $B_{n+1} \subset \underbrace{B_n \times B_n \times \dots \times B_n}_{b_{n+1}/b_n}$ (and hence $X_{n+1} \subset X_n$).

(iv) For any $x, y \in X_n$ $(n \geq 1)$ there is $k \in \mathbb{Z}$ satisfying $\mathrm{dist}(\sigma^k(x), y) \leq 2^{-n}$.

From the condition (iv) it easily follows that the system (X, σ) is minimal. Set $I := \bigcap_{n=0}^{\infty} I_n \subset \mathbb{Z}$. From the condition (i), for $n < m$, $b_n | a_m, b_m$ and hence $b_n \leq b_m - a_m$. So $I \cap \{0,1,2,\dots,b_n-1\} = I_n \cap \{0,1,2,\dots,b_n-1\}$. For each $n \geq 1$,

$$|I_{n+1} \cap \{0,1,2,\dots,b_{n+1}-1\}| = \frac{b_{n+1}-a_{n+1}}{b_n}|I_n \cap \{0,1,2,\dots,b_n-1\}|.$$

Hence

$$|I \cap \{0,1,2,\dots,b_n-1\}| = |I_n \cap \{0,1,2,\dots,b_n-1\}| = b_n \prod_{k=1}^{n}(1-p_k).$$

Lemma 3.1. *Under the above conditions, for any $n \geq 1$, there is a continuous distance-increasing map from $(Y^{Db_n \prod_{k=1}^{n}(1-p_k)}, d_{\ell^\infty})$ to (X, dist_{b_n}).*

Proof. We have $Y^{Db_n \prod_{k=1}^{n}(1-p_k)} = K^{I\cap\{0,1,2,\dots,b_n-1\}}$. Fix a point $p \in K$. We define a map $K^{I\cap\{0,1,2,\dots,b_n-1\}} \to X$ by mapping each $x \in K^{I\cap\{0,1,2,\dots,b_n-1\}}$ to the point $x' \in X$ defined by

$$x'_k := \begin{cases} x_k & (k \in I \cap \{0,1,2,\dots,b_n-1\}) \\ x(m)_{k'} & (\exists m \geq 1, \exists k' \in \{0,1,2,\dots,b_m-1\} \setminus I_m : k' \equiv k \bmod b_m) \\ p & (\text{otherwise}). \end{cases}$$

From the compatibility condition on $\{x(m)\}$, this map is well-defined. We can easily check that this is continuous and distance-increasing. □

Lemma 3.2. $\mathrm{mdim}(X,\sigma) = D/2$.

Proof. We have $\dim B_n = D|I_n \cap \{0,1,2,\dots,b_n-1\}| = Db_n \prod_{k=1}^{n}(1-p_k)$. From Lemma 2.4,

$$\mathrm{mdim}(X,\sigma) \leq \mathrm{mdim}(X_n,\sigma) \leq D\prod_{k=1}^{n}(1-p_k).$$

Letting $n \to \infty$ and using (2), we get the upper bound $\mathrm{mdim}(X,\sigma) \leq D/2$. On the other hand, from Lemma 3.1, for $0 < \varepsilon < 1$,

$$\frac{\mathrm{Widim}_\varepsilon(X, \mathrm{dist}_{b_n})}{b_n} \geq \frac{\mathrm{Widim}_\varepsilon(Y^{Db_n \prod_{k=1}^{n}(1-p_k)}, d_{\ell^\infty})}{b_n} = D\prod_{k=1}^{n}(1-p_k).$$

Here we have used Example 2.1 and the fact that the space $(Y^{Db_n \prod_{k=1}^{n}(1-p_k)}, d_{\ell^\infty})$ contains $([0,1]^{Db_n \prod_{k=1}^{n}(1-p_k)}, d_{\ell^\infty})$. Letting $n \to \infty$ and $\varepsilon \to 0$, we get $\mathrm{mdim}(X,\sigma) \geq D/2$. □

Lemma 3.3. *(X,σ) cannot be embedded into $(([0,1]^D)^{\mathbb{Z}}, \sigma)$.*

Proof. Suppose that there is an embedding f from (X,σ) into $(([0,1]^D)^{\mathbb{Z}}, \sigma)$. Take a distance d' on $([0,1]^D)^{\mathbb{Z}}$. There exists $\varepsilon > 0$ such that if $d'(f(x), f(y)) \leq \varepsilon$ then $\mathrm{dist}(x,y) \leq 1/2$. Since f commutes with the shift transformations, for every $N \geq 1$, if two points $x, y \in X$ satisfy $d'_N(f(x), f(y)) \leq \varepsilon$ then $\mathrm{dist}_N(x,y) \leq 1/2$. We can take a positive integer $L = L(\varepsilon)$ such that if two points $x, y \in ([0,1]^D)^{\mathbb{Z}}$ satisfy $x_n = y_n$ for $-L \leq n \leq L$ then $d'(x,y) \leq \varepsilon$. Then, for every $N \geq 1$, if two points $x, y \in ([0,1]^D)^{\mathbb{Z}}$ satisfy $x_n = y_n$ for $-L \leq n \leq N + L$ then $d'_N(x,y) \leq \varepsilon$.

Let $\pi_{[-L,N+L]} : ([0,1]^D)^{\mathbb{Z}} \to ([0,1]^D)^{\{-L,-L+1,\dots,N+L\}}$ be the natural projection. Then the map $\pi_{[-L,N+L]} \circ f : (X, \mathrm{dist}_N) \to ([0,1]^D)^{\{-L,-L+1,\dots,N+L\}}$ becomes a $1/2$-embedding.

Using Lemma 3.1, we conclude that for any $n \geq 1$ there exists a 1/2-embedding from $(Y^{Db_n \prod_{k=1}^n (1-p_k)}, d_{\ell^\infty})$ to $[0,1]^{D(b_n+2L+1)}$. Then Proposition 2.5 implies

$$D(b_n + 2L + 1) \geq 2Db_n \prod_{k=1}^{n} (1 - p_k),$$

hence $b_n(2\prod_{k=1}^n (1-p_k) - 1) \leq 2L+1$ which for n large contradicts (3). $\square$

The last problem is to define X_n, a_n, b_n and B_n. (Recall that we fixed rational numbers p_n satisfying (2).) We construct them by induction. First we set $X_0 := K^{\mathbb{Z}}$, $a_0 := 0$, $b_0 := 1$, $B_0 := K$. Suppose that we have constructed X_n, a_n, b_n, B_n. Since block-type systems are topologically transitive, there is $\tilde{x} \in X_n$ whose orbit is dense in X_n. We can assume that $\tilde{x}_{lb_n}^{lb_n+b_n-1} \in B_n$ for all integers l.

Take a positive integer L such that for any $x \in X_n$ there is $k \in [-L, L]$ satisfying $\mathrm{dist}(\sigma^k(\tilde{x}), x) \leq 2^{-n-2}$. We take a positive even integer $a_{n+1} > b_n$ sufficiently large so that

- $b_n \mid (a_{n+1}/2)$.
- $a_{n+1} \gg L$. ($a_{n+1} \geq 2L+2n+10$ will do. But the precise estimate is not important.)
- There is a positive integer b_{n+1} such that $b_n < b_{n+1}$, $b_n \mid b_{n+1}$, $p_{n+1} = a_{n+1}/b_{n+1}$, and

$$b_{n+1} \left(2 \prod_{k \leq n+1} (1 - p_k) - 1 \right) \geq n + 1.$$

Then a_{n+1} and b_{n+1} satisfy the condition (i).

We define $B_{n+1} \subset K^{b_{n+1}}$ as the set of $x \in K^{b_{n+1}}$ satisfying

$$x_{lb_n}^{lb_n+b_n-1} \in B_n \ (\forall l \in \mathbb{Z} \text{ with } 0 \leq l < b_{n+1}/b_n), \quad x_{b_{n+1}-a_{n+1}}^{b_{n+1}-1} = \tilde{x}_{-a_{n+1}/2}^{a_{n+1}/2-1}.$$

Here $x_m^n = (x_m, x_{m+1}, \ldots, x_n)$ for $m \leq n$ and $x = (x_0, x_1, \ldots, x_{b_{n+1}-1}) \in K^{b_{n+1}}$. Let X_{n+1} be the block-type system defined by B_{n+1} (condition (ii)). From the definition of $\tilde{x}$ and $a_{n+1} \gg L$, the system X_{n+1} satisfies the condition (iv). We define $x(n+1) \in K^{\{0,1,2,\ldots,b_{n+1}-1\} \setminus I_{n+1}}$ (see the condition (iii)) by

$$x(n+1)_k := \begin{cases} x(n)_{k'} & (\exists k' \in \{0,1,2,\ldots,b_n - 1\} \setminus I_n : k \equiv k' \bmod b_n) \\ \tilde{x}_{k-b_{n+1}+a_{n+1}/2} & (b_{n+1} - a_{n+1} \leq k \leq b_{n+1} - 1). \end{cases}$$

Since we assume $\tilde{x}_{lb_n}^{lb_n+b_n-1} \in B_n$ for all integers l, this is well-defined. (When $n = 0$, we set $x(1)_k = \tilde{x}_{k-b_1+a_1/2}$ for $b_1 - a_1 \leq k \leq b_1 - 1$.) We can easily check that the condition (iii) is satisfied. This completes the proof of Theorem 1.3.

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