1 From groups to graphs

Setting: G, H, X. G a group, H a subgroup; X a set of generators for H. Often we assume X is *symmetric*: $1 \in X, X = X^{-1}$.

We have an equivalence relation $\mathbf{H}$, the (left) coset equivalence relation; and a graph $\mathbf{X}$, the Cayley graph. $\mathbf{H}$ is generated by $\mathbf{X}$.

Transpose to: G a set; H an equivalence relation. $X \subset G^2$ a generating set. *Symmetric:* the diagonal on G is contained in X; and $X = X^t$. So $H = \cup X \circ \cdots \circ X$.

In both cases, an associated metric $d = d_X$, generated by: $d(x, y) \leq 1$ if $(x, y) \in X$. If we are given a family X_i, let $d(x, y) \leq n$ iff there exist $x = x_1 \ldots x_n = y$ with $(x_k, x_{k+1}) \in \cup X_i$.

Definition 1.1. *A symmetric $X \subset G^2$ is a k-approximate equivalence relation if the valency is of a fixed order of magnitude $|X(a)| \leq k|X(b)|$ for all $a, b \in G$, and every 2-ball is a union of k 1-balls.*

- *Say two metrics d, d' are k-commensurable at scale α if an α-ball of d' is contained in $\leq k$ balls of d-radius α, and vice versa.*
- *A metric space is k-doubling at scale α if $d, (1/2)d$ are k-commensurable at scale α.*
- *Thus: for a k- approximate equivalence relation X, d_X is k-doubling at scale 1.*
- *X is an approximate subgroup of G iff $\mathbf{X}$ is an approximate equivalence relation on G.*

Theorem 1.2 (strong approximation: groups)**.** *Let $F = \mathbb{F}_p$, p nonstandard. Let $G = GL_n(F)$, and let X_i be a family of definable subsets. Then there exists a definable H such that:*

1. *H is a subgroup of G*
2. *$H \subset < \cup X_i >$.*
3. *X_i/H is finite (bounded.)*

Moreover, 'definable' here can be made explicit as follows: there exists a homomorphism of algebraic groups (with bounded data), with finite kernel

$$h : \widetilde{H} \to G$$

such that $H = h(\widetilde{H}(F))$.

Applied to the family of one-dimensional unipotent subgroups X_i of *an arbitrary subgroup* Γ *of* G, this shows that Γ contains a *definable* normal subgroup H with Γ/H Abelian-by-bounded. Thus the image of a Zariski dense subgroup of $SL_n(\mathbb{F}_p)$ has bounded index, and is as above. (Weisfeiler 1984, Nori 1987, Gabber, precedents by Eichler 1938, Kneser 1966,...).

Theorem 1.3 (strong approximation: graphs)**.** *Let $F = \mathbb{F}_p$, p nonstandard, G definable over F. Let $X \subset G^2$ be definable. Then there exists $m, m' \in \mathbb{N}$ and a 0-definable $H \leq G^2$ such that:*

1. *H is an equivalence relation on G*
2. *If $(a, b) \in H$ then $d_X(a, b) \leq m$.*
3. *X/H has valency (degree) $\leq m'$.*

Moreover, H is algebraic: *$(a, b) \in H$ iff $\phi(h^{-1}(a)) = \phi(h^{-1}(b))$ for some morphism of varieties $h : \widetilde{G} \to G$ with finite fibers, and regular functions ϕ on $\widetilde{G}$, with $\phi \circ h^{-1}$ well-defined.*

Generate as long as dimension increases; then show there are no definable approximate equivalence relations.

If stated for standard primes: the bounds m, m' on valency and on diameter are independent of p; H varies through only finitely many possible definitions (given G, X); and the complexity of $h, \widetilde{G}, \phi$ is bounded independently of p.

Example 1.4. Fix an algebraic group G_0, e.g. $G_0 = SL_k$. let $F = GF(p)$, let Ω_1, Ω_2 be unipotent orbits in $G_0(F)$, and make $\Gamma = \Omega_1 \times \Omega_2$ into a graph by letting (a, b) be adjacent to $(a, a^{-1}ba)$ and to $(b^{-1}ab, b)$.

Invariants of the connected components:
the group $< a, b >$ generated by (a, b). The fact that this is a definable invariant is the strong approximation lemma for groups!

The trace $tr(ab)$.

Further algebraic invariants.

For pairs (a, b) that generate SL_k, I do not know if further invariants are needed; Gamburd and Sarnak have results on related graphs.

2 Stabilizers

Theorem 2.1 (H. ,Sanders 2009)**.** *Let X be a k-approximate group. Then there exists Y with $Y^{\cdot 8} \subset X^{\cdot 4}$, X contained in boundedly many cosets of Y.*

Theorem 2.2. *Let X be a k- approximate equivalence relation on G. Then there exists $S \subset G^2$ such that $S^{\circ 8} \subset X^{\circ 4}$, and for all $a \in \Omega$ outside an ϵ-slice U, $|S(a)| \geq O_k(1)|X(a)|$.*

Moreover S is 0-definable, uniformly in (Ω, X), in an appropriate logic; in particular $Aut(\Omega, X)$ leaves U, S invariant.

The invariance implies the group-theoretic version.

The 0-definability of S will be essential, in moving from the approximate symmetry of a graph to automorphisms of an associated locally compact space.

Closely related to Lovász-Szegedy *graphons* in cases of bounded diameter at least. However, to know that the definition of $W \circ W$ agrees with the one we give on elements, we need to know essentially the independence theorem; so it does not appear to give a new proof of the stabilizer theorem.

3 Approximate symmetry

A distance between finite graphs: (Keisler-Hoover, Gromov (measured metric spaes), Benjamini-Schramm)

Definition 3.1.

$$\rho(\Omega, \Omega') = \sup\{\frac{1}{m} : (\exists \Gamma)|\Gamma| = m, |Pr(\Gamma, \Omega) - Pr(\Gamma, \Omega')| \geq \frac{1}{m}\}$$

Where $Pr(\Gamma, \Omega) = |Hom(\Gamma, \Omega)|/\Omega^m$.

A similar definition applies to *pointed graphs*.

Definition 3.2. (Ω, X) *is* ϵ-homogeneous *if $\rho_{ptd}((\Omega, a), (\Omega, b)) \leq \epsilon$ for all $a, b \in \Omega$.*

Definition 3.3. $\Omega_n \to \Omega$ *if $\rho_{ptd}(\Omega_n, a_n), (\Omega, a) \to 0$ for all $a_n \in \Omega_n, a \in \Omega$.*

Of course, this only makes sense if the Ω_n are increasingly ϵ-homogeneous.

We will really have a stronger notion of convergence: there will be a metric d on Ω_n, $d(x, y) = 1$ iff $(x, y) \in X$, such that Gromov-Hausdorff convergence holds with respect to the metrics.

A Riemannian homogeneous space is a Riemannian manifold, with transitive isometry group (Classified by Wolf when the stabilizer acts irreducibly on tangent space.)

A Riemannian model is a Riemannian homogeneous space, with compact point stabilizer, and with the approximate equivalence relation: $d(x, y) \leq 1$.

Riemannian models have ϵ-homogeneous approximations for any ϵ. Let G/K be a Riemannian homogeneous space; G a Lie group, K compact. Let Λ be a lattice of large covolume. Let n be large, and choose n points at random on $\Lambda \backslash G/K$.

Theorem 3.4. *Let (G_n, X_n) an approximately homogeneous sequence of approximate equivalence relations. Then some subsequence approaches a limit (Γ, X), admitting a homomorphism to a vertex transtive graph B of bounded degree, such that each fiber is commensurable to a Riemannian model.*

4 Partial Bourgain systems

Theorem 4.1. *Fix $k \in \mathbb{N}$. Then there exists $e^* \in \mathbb{N}$ such that the following holds: Let G be any group, X a finite subset, and assume $|XX^{-1}X| \leq k|X|$.*

Then there are $2 \leq e, c \leq e^*$, and $N > 2^{2^{2^{ec}}}$ subsets $X_N \subseteq X_{N-1} \subseteq \cdots \subseteq X_1 \subseteq X^{-1}XX^{-1}X$ such that X, X_1 are e-commensurable, and for $1 \leq m, n < N$ we have:

1. $X_n = X_n^{-1}$
2. $X_{n+1}X_{n+1} \subseteq X_n$
3. *X_n is contained in the union of c translates of X_{n+1}.*
4. *$[X_n, X_m] \subseteq X_k$ whenever $k \leq N$ and $k < n + m$.*

Theorem 4.2. *Fix $k \in \mathbb{N}$. Then there exists $e^* \in \mathbb{N}$ and $\epsilon > 0$ such that the following holds: Let (G, X) be an ϵ-homogeneous approximate equivalence relation. Then there are $e, c \leq e^*$ and $N > 2^{2^{2^{eck}}}$, and a metric d_N on X such that:*

X-balls are covered by $\leq e$ d_N-balls of radius 1, while d_N-balls of radius 1 are contained in X-balls of radius 4. (and so in k^4 X-balls of radius 1); and for $1 \leq m, n < N$, d_N is c- doubling at scale 2^{-n}, i.e. d_N-balls of radius 2^n are contained in c balls of radius 2^{n+1}.

Problem 4.3. *Complete this with an analogue of (4).*

Problem 4.4. *Exploit homogeneity on* types *to obtain a statement without the approximate homogeneity assumption. (embedding of sections into a Riemannian homogeneous space.)*

5 Proof of stablizer lemma

- xS_ny iff $\mu\{z : |\mu(R(x) \cap R(z)) - \mu(R(y) \cap R(z))| \geq 2^{-n}\} \leq 2^{-n}\}$
- At limit, $\cap_n S_n$: for almost all z, $\mu(R(x) \cap R(z)) = \mu(R(y) \cap R(z))$. It is cobounded.

- $S_{n+1} \circ S_{n+1} \subset S_n$. (Away from measure 0).
- $S_n \subset R^{\circ 4}$, for large n.
- S_n is definable in terms of R using *probability logic*. This definability will be essential, showing that (approximate) symmetries of the graph, are (approximate) symmetries of the associated refining metric.
- The proof uses stability: $\mu(R(x) \cap R(z))$ is a stable real-valued formula. New proofs of this by Tao.

6 Approximately homogeneus approximate equivalence relations (proof)

- Ultraproduct. Obtain two equivalence relations: $\widetilde{E}$ = finite distance. Γ = infinitesimal distance.
- Let Ω be a class of $\widetilde{E}$; then Ω/Γ is locally compact.
- $G := Aut(\Omega/\Gamma)$ acts transitively on Ω, by isometries of the fine metric. Keisler,Gromov-Vershik,
- A locally compact structure on G (compact-open topology.) The stabilizer of a point is compact.
- By Gleason-Yamabe, an open subgroup H, a small normal compact subgroup N, with H/N a Lie group.
- From Ω to an H-orbit: locally bounded distortion. (R induces a graph of bounded degree on Ω/H.)
- Factor out N. Obtain a coarser equivalence relation than the original distance-zero, but still contained in $d_R \leq 4$.
- Now the Lie group H/N acts transitively on Ω/Γ, compact point stabilizer. Find an invariant Riemannian metric. This metric is doubling up to distance 1, and the distance 1 relation is commensurable with d_R.
- For partial Bourgain systems: return information to finite factors, up to scale $\Psi(c)$.

7 Comparison

Theorem 7.1 (Benjamini- Finucane-Tessera 2012). *1. Let (X_n) be an unbounded sequence of finite, connected, vertex transitive graphs with bounded degree such that $|X_n| = o(diam(X_n)^q)$ for some $q > 0$. After rescaling by the diameter, some subsequence converges in the Gromov Hausdorff distance to a torus of dimension $< q$, with an invariant metric.*

2. If q is close to 1*, then the scaling limit of (X_n) is S^1*, even if X_n is only roughly transitive