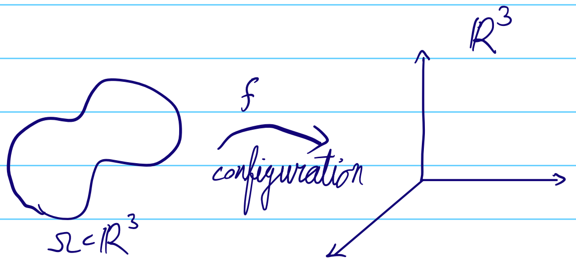
Introduction & motivation

Standard hyperelasticity



Elastic energy: $E(f)=\int_\Omega W(Df_x)\,dx$

- $W:\mathbb{R}^{3\times 3}\to[0,\infty)$, $\{W=0\}=SO(3)$ (single well structure)
- $W(RA)=W(A)$ $\forall A\in\mathbb{R}^{3\times3}$, $R\in SO(3)$ (frame indifference)
- $W(A)\geq c\,\mathrm{dist}^2(A,SO(3))$ (coercivity)

$\vdots$

Isotropy group of W $G_W=\{Q\mid W(AQ)=W(A)\ \forall A\}<SO(3)$

Isotropic material $G_W=SO(3)$

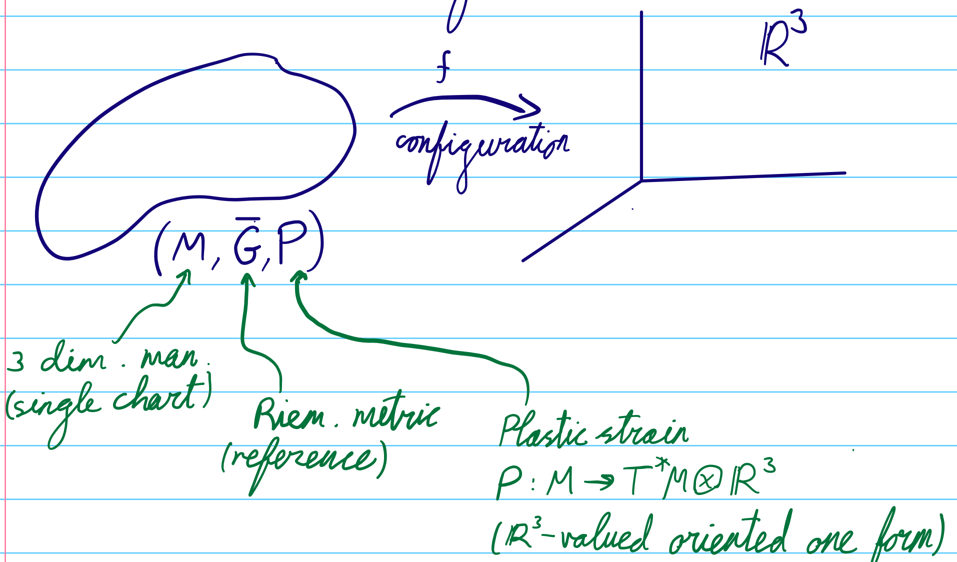
Example: $W(A)=\mathrm{dist}^2(A,SO(3))$

The well assumption $\{W=0\}=SO(3)$ implies the existence of zero energy (reference) configurations: $f(x)=Rx+b$.

Main focus of this course:

Problems in which no reference configurations exist

I Non-Euclidean elasticity



Elastic energy: $\underset{\overset{\shortparallel}{E_{M,P}(f)}}{E_M(f)} = \int_M W(Df_x \circ P_x^{-1})\, dV_{\bar G}$

Volume form of $\bar G$
$|\det \bar G|^{\frac{1}{2}} dx$

"implant map"

Compatibility of P & $\bar G$: $\bar G_x(v, w) = \langle P_x v, P_x w \rangle$, hence $P \rightsquigarrow \bar G$.

I.e. $P_x \in SO(\bar G_x) := \{ A : T_x M \to \mathbb{R}^3 \mid A \text{ is orientation-preserving isometry} \}$

Example: In a chart, $\sqrt{\bar G}$ is a compatible strain.

If P, P' are two plastic strains compatible with $\bar G$, then $P'_x = Q_x P_x$ for some $Q_x \in SO(3)$.

$$\forall_x \; Q_x \in G_W \quad \Longrightarrow \quad E_{M,P} = E_{M,P'}$$

In particular, if W is isotropic, then

$$E_M(f) = \int W(Df_x \circ \bar G_x^{-\frac{1}{2}})\, |\det \bar G_x|^{\frac{1}{2}} dx$$

Corollary: $E_M(f) = 0 \iff Df_x \in SO(\bar G_x)$ a.e.

Pf: $W(Df_x \circ P_x^{-1}) = 0 \iff Df_x \circ P_x^{-1} \in SO(3)$

$$\iff Df_x \in SO(3) P_x = SO(\bar G_x)$$

Laplace Beltrami
$\frac{1}{|\bar{G}|^{\frac{1}{2}}}\partial_j\left(\bar{G}^{jk}|\bar{G}|^{\frac{1}{2}}\partial_k\right)$

Prop: $Df_x \in SO(\bar{G}_x)$ a.e. $\Rightarrow \Delta_{\bar{G}} f = 0$.
In particular, f is smooth.
This implies that $f: M \to \mathbb{R}^3$ is an isometric imm., hence $\bar{G}$ is flat $- \mathcal{R}^{\bar{G}} \equiv 0$.

$\det_{\bar{G}} A := |\bar{G}|^{-\frac{1}{2}} \det A$
$\operatorname{cof}_{\bar{G}} A := |\bar{G}|^{-\frac{1}{2}} \operatorname{cof} A \bullet \bar{G}$

Pf: $A \in SO(\bar{G}_x) \Leftrightarrow \det_{\bar{G}} A = 1$, $\operatorname{cof}_{\bar{G}} A = A$

$A^{\#} = A\,\bar{G}^{-1}$
$\operatorname{div}_{\bar{G}} V = |\bar{G}|^{-\frac{1}{2}} \partial_j\left(|\bar{G}|^{\frac{1}{2}} V^j\right)$

As in Euclidean case, $\operatorname{div}_{\bar{G}}(\operatorname{cof}_{\bar{G}} Df)^{\#} = 0 \quad \forall f$

Under our assumption $\operatorname{cof}_{\bar{G}} Df = Df$, hence
$\Delta_{\bar{G}} f = \operatorname{div}_{\bar{G}}(Df)^{\#} = 0$. (can also be written using a weak for.)

The flatness follows from Riemann's theorem.

Thm (LP'11, KMS'19): $\inf E_M = 0 \Rightarrow \mathcal{R}^{\bar{G}} \equiv 0$. The converse holds if M is simply-connected.

In KMS'19 also for non-Eucl. target

These works show a stronger claim: $\inf E_M = 0 \Leftrightarrow \exists f \in C^\infty(M;\mathbb{R}^n)$ iso. imm.

Pf: $\Rightarrow$ We will deduce it from a following estimate.
$\Leftarrow$ flatness + simply-conn. $\rightsquigarrow \exists f: M \to \mathbb{R}^3$ iso. imm. $\Rightarrow E_M(f) = 0$.

So for M simply-conn. we get $\inf E_M > 0 \Leftrightarrow \mathcal{R}^{\bar{G}} \not\equiv 0$. Can we get more quantitative?
Let's "localize" the question:

Thm (MS'19, KM'21): Let $p \in M$, then $\inf E_{B_r(p)} = r^4 |\mathcal{R}^{\bar{G}}(p)|^2 + o(r^4)$.

some norm, ind. of P for isotropic W.

Pf: For simplicity, choose $W(A) = \operatorname{dist}^2(A, SO(n))$.
We can choose normal coordinates on $B_r(p)$ (for small enough $r>0$), and then

$$\begin{aligned} B_r(p) &\rightsquigarrow B_r(0) \subset \mathbb{R}^n, \qquad \bar{G}(x)_{ij} = \delta_{ij} + \tfrac{1}{3}\mathcal{R}_{kijl}(p)\,x^k x^l + O(|x|^3) \\ q &\longmapsto x \end{aligned}$$

hence $\bar{G}^{-\frac{1}{2}}(x)_i^j = \delta_i^j \underbrace{- \tfrac{1}{6} R_{kijl}(p)\,x^k x^l}_{T(x)} + O(|x|^3)$
and thus, for the map $\iota(x) = x$ we have

our energy scaling

$$E_{B_r(p)}(\iota) = \fint_{B_r(0)} \operatorname{dist}^2\left(I - \tfrac{1}{6}R_{kijl}(p)\,x^k x^l + O(|x|^3)\right) \cdot \left(1 + O(|x|^2)\right) dx \sim r^4$$

Lower bnd: Assume $E_{B_r(p)}(f_r) \leq Cr^4$. Then, since $|\bar{G}^{-\frac{1}{2}} - I| \leq C|x|^2$,

$$\operatorname{dist}(f_r, SO(n)) \leq \operatorname{dist}(f_r \circ \bar{G}^{-\frac{1}{2}}, SO(n)) + C|f_r||x|^2$$

$$\leq C\left(\operatorname{dist}(f_r \circ \bar{G}^{-\frac{1}{2}}, SO(n)) + |x|^2\right)$$

hence $\int_{B_r(0)} \operatorname{dist}^2(f_r, SO(n))\, dx = O(r^4)$.

From FJM we have $R \in SO(n)$ s.t. $\int_{B_r(0)} |Df_r - R|^2 \leq Cr^4$.

$\exists c_r$ s.t. $\bar{f}_r = R^T f_r - c_r$ sat. $\int |\bar{f}_r - x|^2 + |D\bar{f}_r - I|^2 \leq Cr^4$

Define now $y = \frac{x}{r} \in B_1(0)$, $v_r(y) = \bar{f}_r(ry) - ry$, then

$$\frac{1}{r^3} Dv_r \longrightarrow Df \quad \text{in } L^2(B_1(0), \mathbb{R}^n)$$

$$D\bar{f}_r(ry) \circ \bar{G}^{-\frac{1}{2}}(ry) = I + \underbrace{\frac{1}{r^3} Dv_r(y)}_{\xrightarrow{L^2} Df} + \underbrace{\frac{\bar{G}^{-\frac{1}{2}}(ry) - I}{r^2}}_{\xrightarrow{L^\infty} -T(y)} + \underbrace{\frac{1}{r^3} Dv_r(y)\left(\bar{G}^{-\frac{1}{2}}(ry) - I\right)}_{\xrightarrow{L^2} 0}$$

Hence:

$$\liminf \frac{1}{r^4} E_{B_r(p)}(f_r) = \liminf \frac{1}{r^4} E_{B_r(p)}(\bar{f}_r)$$

$$= \liminf \frac{1}{r^4} \int_{\substack{B_r(0) \\ B_1(0)}} \operatorname{dist}^2\left(Df_r(\overset{ry}{x}) \circ \bar{G}^{-\frac{1}{2}}(\overset{ry}{x})\right)\left(1 + O(\overset{r^2}{|x|^2})\right) \overset{dy}{dx}$$

$$\operatorname{dist}(I + A, SO(n)) = |\operatorname{sym} A| + O(|A|^2) \leftarrow$$

$$\geq \int_{B_1(0)} |\operatorname{sym} Df - T|^2\, dy \geq \min_{f \in W^{1,2}} \int_{B_1(0)} |\operatorname{sym} Df - T|^2\, dy =: I(\mathcal{R}(p))$$

Now, $\mathcal{R} \mapsto (I(\mathcal{R}))^{\frac{1}{2}}$ is a norm:

homogeneity + triangle ineq. are easy.

Positivity follows from Saint-Venant compatibility condition:

$$T = \mathrm{sym}\, Df \Leftrightarrow \underbrace{\partial_{ij}T_{kl} + \partial_{kl}T_{ij} - \partial_{ik}T_{jl} - \partial_{jl}T_{ik}}_{R_{iklj}^{(p)}} = 0$$

<u>Upper bound</u>: Let f be a min. of $\int_{B_1(0)} |\mathrm{sym}\, Df - T|^2 dy$

and define $f_r(x) = x + r^3 f(x/r)$.

The above calculation yields $\frac{1}{r^4} E_{B_r(p)}(f_r) \longrightarrow I(R)$ ▨

<u>Comment</u>: If we take another isotropic W, we get $\int Q(\mathrm{sym}\, Df - T)$,

where $Q(A) = \frac{1}{2} D_I^2 W(A,A)$ is positive def. on sym. matrices.

If W is not isotropic, then, writing $P_x = \underset{SO(n)}{\underset{\cap}{Q_x}} G^{-\frac{1}{2}}$, we get

$$\int_{B_1(0)} Q(Q_p^T(\mathrm{sym}\, Df - T)Q_p)\, dy$$

<u>Open question</u>: Finite balls — if $\tilde{M}_{K_0,r_0}$ is a geodesic ball of radius r_0 and const. curvature $K_0 > 0$, and M is a " " " " r_0 " sectional curv. $\geq K_0$, do we have

$$\inf E_{\tilde{M}_{K_0,r_0}} \leq \inf E_M \ ?$$

II Defects in solids

We saw inf $E_\mu = 0 \Rightarrow R^{\bar G} = 0$.
The converse is true only for simply-connected domains:
Topology is also an obstruction!

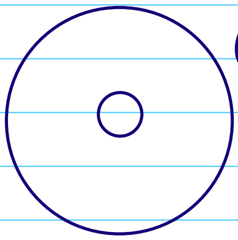
Focus on locally-flat ($R^{\bar G} \equiv 0$), multiply-connected bodies.

Volterra cut-and-weld constructions - Show picture

$R^{\bar G} \equiv 0 \Rightarrow M$ is locally isometrically embeddable in $\mathbb{R}^n$

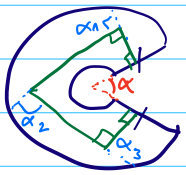
$\Rightarrow$ Notions s.a. straight lines, geodesic curv. etc. are as in $\mathbb{R}^n$.
(everything that is measured locally)
In part., locally path-ind. par. transport (identifying vectors in different tangent spaces)

Defects in 2D



$(M,\bar G)$, $\bar G$ locally flat.

Disclination

- 

cannot be isom. imm. in $\mathbb{R}^2$:
The sum of angles in any triangle $= \pi + \alpha > \pi$
(preserved under isometry)

Gauss-Bonnet: for any triangle in a simply-connected surface

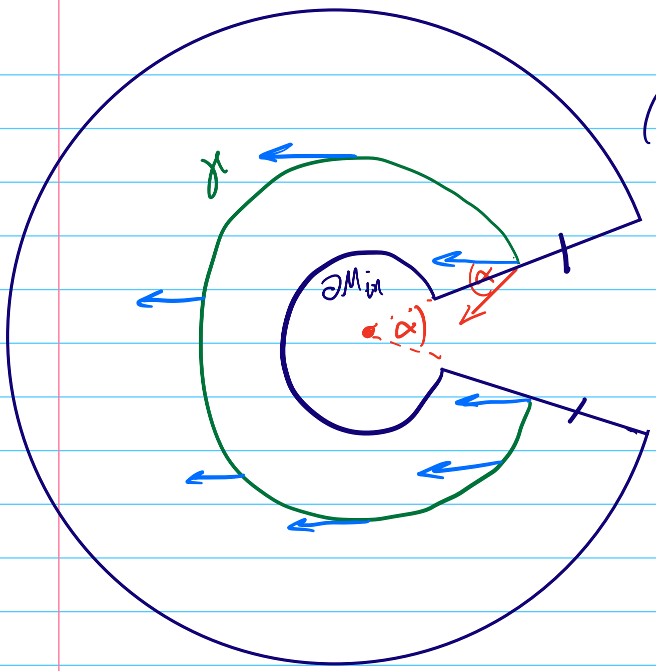
$$\iint K_{\bar G} = 2\pi - \sum \alpha_i - \int \kappa_g \, ds$$

if time permits

in our case: $2\pi - \overbrace{(2\pi-\alpha)} - \underbrace{\quad}_{0} = \alpha$

hence we can think of the "hole" having total curvature α.
Given $R^{\bar G} \equiv 0$, we can measure this defect by calculating $2\pi - \int \kappa_g \, ds$ on any smooth closed curve around the core

Disclination = curvature charge



$(M,\bar{G})$ cannot be iso. imm. in $\mathbb{R}^2$:

If we par. trans. a vector along γ (homotopic to ∂M_{in}) we get a rotation of α!

If we do it in a closed curve in $\mathbb{R}^2$ we get zero.

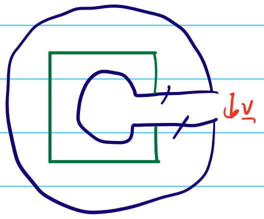
$\mathrm{Hol}_\gamma = R_\alpha$

Cor: No parallel vector fields!

Comment: In a general manifold $(M,\bar{G})$, one can show that the holonomy of an infinitesimally small curve converges to the Riem. curv. at the point.

Disclination = curvature charge!

(edge)
Dislocation

- 

No curvature charge (Gauss-Bonnet is satisfied, $\mathrm{Hol}_\gamma = I \ \forall \gamma$)

Cannot be isometrically immersed in $\mathbb{R}^2$:
In the rectangle, opposite edges are not equal.

More generally, any simple, oriented closed curve γ around the core satisfies

$$\oint \Gamma_{\gamma(t)}^{\gamma(0)} \dot{\gamma}(t)\,dt = b(\gamma(0)) \in T_{\gamma(0)}M$$

Par. trans. along γ

Burgers vector (par. vector field)

The **Burgers vector** is the charge of the dislocation.

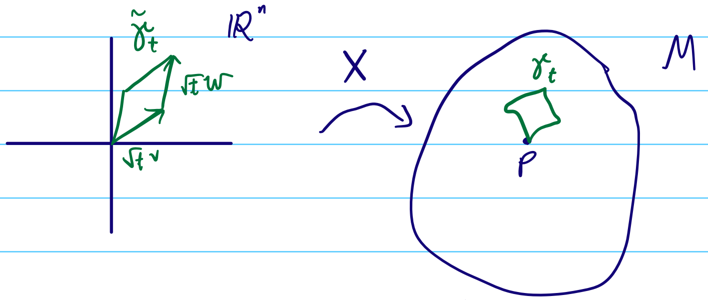
If time permits

In Riem. geo., we can define many connections, $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k$.

The **torsion** of the connection is $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$

$$T_{ij} = (\Gamma_{ij}^k - \Gamma_{ji}^k)\partial_k$$

Thm: Given a connection ∇, and let $X : U \subset M \to \mathbb{R}^n$ be a coordinate chart, with $X(p) = 0$, and let $v, w \in T_pM$, $\tilde{v} = dX(v)$, $\tilde{w} = dX(w)$

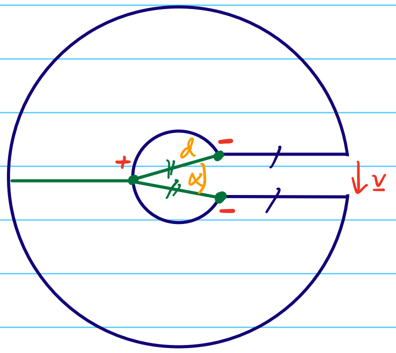


Then, $T_p(v,w) = \frac{d}{dt}\Big|_{t=0} \oint \underbrace{\Gamma^p_{\gamma_t(s)}}_{\text{par. trans. along } \gamma} \dot{\gamma}_t(s)\,ds$.

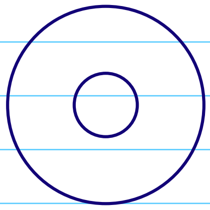
Dislocation = torsion charge.

Remark: If the core contains a disclination, then the Burgers vector is path-dependent, i.e., not well-defined.

"Edge dislocation = Curvature dipole"



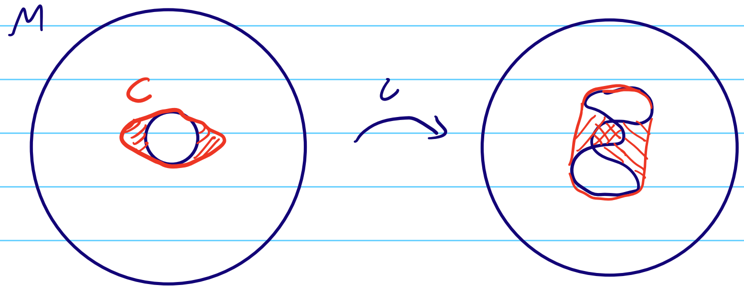
$|\underline{v}| = 2d\sin\frac{\alpha}{2}$

-  $(M, \bar{G})$, $\bar{G}$ with no $\overset{\mathrm{Hol}_\gamma = 0}{\text{curvature}}$ and $\overset{b=0}{\text{torsion}}$ charges.

  Can we embed it iso. in $\mathbb{R}^2$?

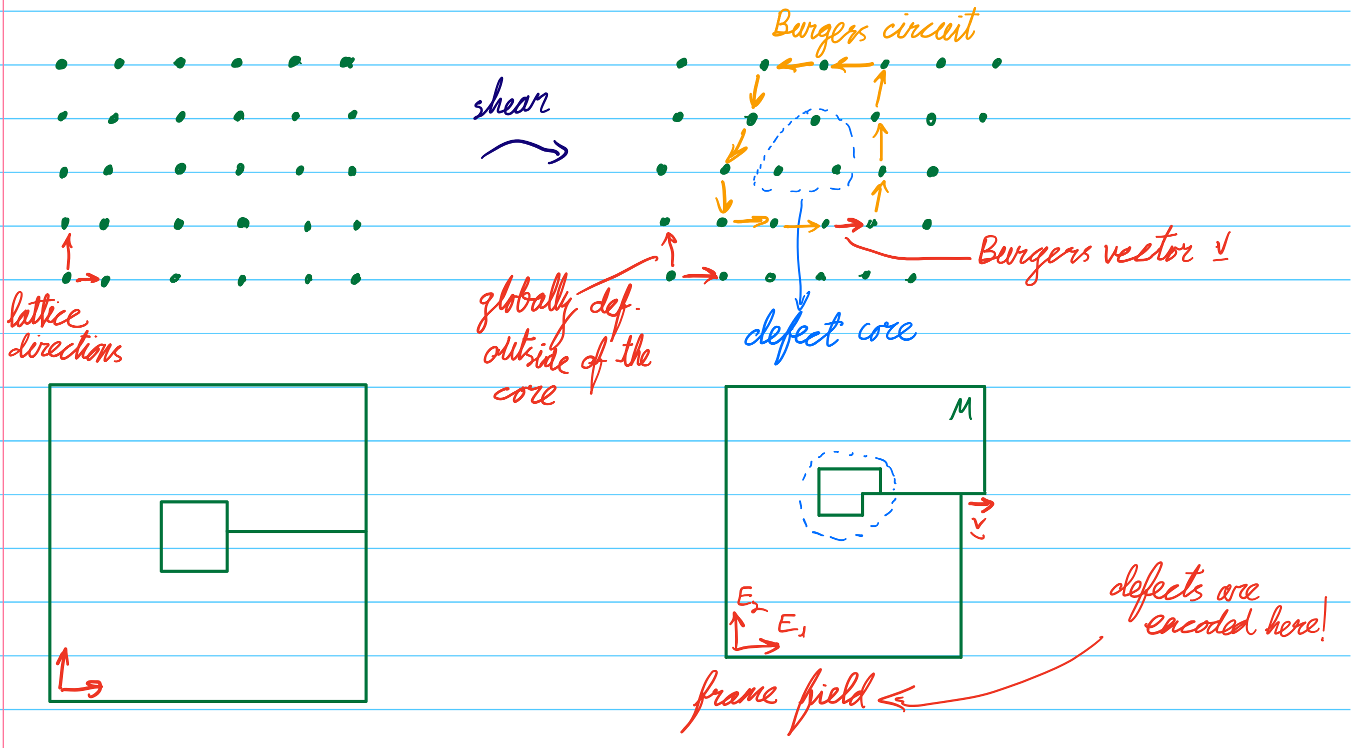
Thm $\overset{\text{KM'23}}{\text{(KMS'15)}}$: Let $\partial M_{in} \subset M$ be the inner boundary, and let $C = \mathrm{conv}(\partial M_{in})$ its geodesic convex hull.
Then, under the above assumptions, $(M \setminus C, \bar{G})$ can be iso. emb. in $\mathbb{R}^2$.

Ex:



$\boxed{\text{From now on — focus on dislocations}}$ (following mostly Kupferman-M. '23)

Dislocation in lattices:



$\{E_1, E_2\}$ frame-field $\rightsquigarrow$ $E : \mathbb{R}^2 \to TM$ frame $\rightsquigarrow$ $P : TM \to \mathbb{R}^2$ dual frame

Dislocations in $(M, \bar{G}, P)$:

① $dP = 0 \implies R^{\bar{G}} \equiv 0$, no curvature charge in core

Pf: $P = df$ locally, hence $\bar{G}(X,Y) = \langle df(X), df(Y) \rangle$ locally $\Rightarrow R^{G} \equiv 0$.

Again, locally, $\nabla^{\bar{G}} P = \nabla^{f^*\delta} df = df(\nabla^{\delta} \mathrm{Id}) = 0$ hence P is a global, parallel section of $SO(\bar{G})$. $\implies$ no holonomy $\implies$ no dislocation.

② $\int_{\gamma} P = \int_0^1 P_{\gamma(t)}(\dot{\gamma}(t))\, dt = \underline{v}$ Burgers vector. $P(\underline{b}) = \underline{v}$.

↪ curve homotopic to inner bdry.

Def: A two-dim. body w. an edge dislocation is $(M, \bar{G}, P)$, $M \overset{diff.}{\cong} \mathbb{R}^2 \setminus B_1$,
P satisfies ①, ②,
∂M has winding number 1. $\left(\frac{1}{2\pi} \int_{\gamma} \kappa(s)\, ds = 1 \right)$

Ex: $M = \{(r,\varphi) \mid r \geq |\underline{v}|\}$

- $\hat{P}_{\underline{v}} = I + \frac{\underline{v}}{2\pi} d\varphi \rightsquigarrow \bar{G} = \begin{pmatrix} 1 & \frac{1}{\pi}(v_1 \cos\varphi + v_2 \sin\varphi) \\ & \left(r + \frac{1}{2\pi}(-v_1 \sin\varphi + v_2 \cos\varphi)\right)^2 \end{pmatrix}$

- $\bar{G} = \begin{pmatrix} 1 & 0 \\ & \left(r + \frac{|\underline{v}|}{\pi} \cos(\varphi - \varphi_0)\right)^2 \end{pmatrix}$

- $\bar{G} = \exp\left(\frac{|\underline{v}| \cos(\varphi - \varphi_0)}{\pi r}\right) \begin{pmatrix} 1 & 0 \\ & r^2 \end{pmatrix}$

Q: Are they isometric?

Thm (Kupferman-M.): If $(M, \bar{G}, P)$ is a complete body with an edge dislocation, then $(M \setminus \operatorname{conv}(\partial M_{in}), \bar{G}, P)$ is obtained by a Volterra construction.

Cor: $\underline{v}$ uniquely determines a body w. edge disloc. up to the shape of the core.

Energy scaling of a single dislocation

Consider $M^R_{\underline{v}} = \{(r,\varphi) \mid r \in [|\underline{v}|, R]\}$, $\hat{P}_{\underline{v}} = I + \frac{\underline{v}}{2\pi} d\varphi$

$\hat{g}_{\underline{v}}$ induced by $\hat{P}_{\underline{v}}$.

Lemma: $\frac{1}{C}\delta < \hat{g}_v < C\delta$, $\frac{1}{C} < \left|\frac{dV_{\hat{g}}}{dx}\right| < C$, C ind. of $\underline{v}$

Prop: Let $\delta \in (0,1)$ s.t. $\delta R \in [|\underline{v}|, R]$. Then

$$\inf_{H^1(M_{\underline{v}};\mathbb{R}^2)} \int_{M^R_{\underline{v}} \setminus M^{\delta R}_{\underline{v}}} \mathrm{dist}^2(Df, SO(\hat{g}_v))\, dV_{\hat{g}_v} \sim |\underline{v}|^2 \log\frac{1}{\delta}$$

In particular, for $\delta R = |\underline{v}|$ we get

$$\inf_{H^1(M_{\underline{v}};\mathbb{R}^2)} \int_{M^R_{\underline{v}}} \mathrm{dist}^2(Df, SO(\hat{g}_v))\, dV_{\hat{g}_v} \sim |\underline{v}|^2 \log\frac{R}{|\underline{v}|}$$

Upper bnd: Choose $f(\overset{\overset{M_{\underline{v}}}{\cup}}{(r,\varphi)}) = \overset{\overset{\mathbb{R}^2}{\cup}}{(r,\varphi)}$, and then

$$\mathrm{dist}(I, SO(\hat{g}_v)) \le |I - \hat{P}_{\underline{v}}| = \frac{|\underline{v}|}{2\pi}\frac{1}{r}$$

(we're lying here a bit since dist & $|\cdot|$ are w.r.t. $\hat{g}_v$ this calculation is w.r.t. Euclidean norm, but they are uniformly equivalent, ind. of $\underline{v}$)

Thus, $\inf \int \dots \lesssim \int_\delta^R |\underline{v}|^2 \frac{1}{r^2} r\, dr = |\underline{v}|^2 \log\frac{1}{\delta}$.

Lower bnd:

FJM for half annuli: $\Omega = \{(r,\varphi) \mid r\in(R_1,R_2), \varphi\in(0,\pi)\}$, $\frac{R_2}{R_1}\geq\frac{3}{2}$.

$\exists C>0$ (ind. of R_1,R_2) s.t. $\forall f\in H^1(\Omega;\mathbb{R}^2)$ $\exists U\in SO(2)$ s.t.

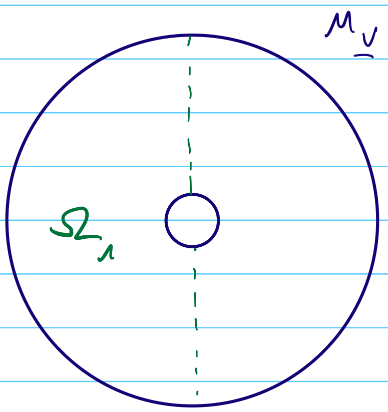
$$\int_\Omega |Df-U|^2\,dx \leq C\int_\Omega \mathrm{dist}^2(Df,SO(2))\,dx$$

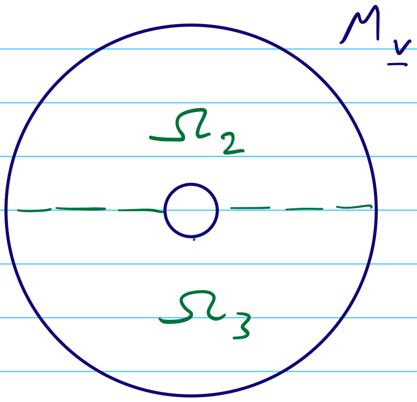
FJM for $\hat{M}_{\underline{v}}$: Assume $R>10|\underline{v}|$, $\delta>0$ s.t. $\delta R\in[|\underline{v}|,\frac{2}{3}R)$

$\exists C>0$ ind. of $|\underline{v}|, R, \delta$ s.t. $\forall f\in H^1(\hat{M}^R_{\underline{v}};\mathbb{R}^2)$, $\exists U\in SO(2)$ s.t.

$$\int_{\hat{M}^R_{\underline{v}}\setminus\hat{M}^\delta_{\underline{v}}} |Df-U\hat{P}_{\underline{v}}|^2\,dV_{\hat{g}_v} \leq C\int_{\hat{M}^R_{\underline{v}}\setminus\hat{M}^\delta_{\underline{v}}} \mathrm{dist}^2(Df,SO(\hat{g}_v))\,dV_{\hat{g}_v}$$

Pf:





Ω_i simply connected flat manifold, hence

$$\begin{tikzcd}
(\Omega_i, \hat{g}_v, \hat{P}_{\underline{v}}) \arrow[r, "\iota", "\text{(isometry)}"'] & (\Omega, \delta, P_0) \\
f \arrow[r, squiggly] & \iota\circ f \arrow[d, "\text{FJM}"] \\
|Df-U\hat{P}_{\underline{v}}|^2_{g_v}\leq C\cdots & |D(\iota\circ f)-UP_0|^2\leq C\cdots \arrow[l]
\end{tikzcd}$$

$\subseteq\mathbb{R}^2$ uni. biLip. equiv. to half annulus

$P_0 \in SO(2)$ const. since $dP_0=0$

Pf of lower bnd:

$$\int_{M_{\underline{v}}^R \setminus M_{\underline{v}}^\delta} \mathrm{dist}^2(Df, SO(\mathring{g}_v))\, dV_{\hat{g}_{\underline{v}}} \gtrsim \int |Df - U\hat{P}_{\underline{v}}|^2_{\hat{g}_v}\, dV_{\hat{g}}$$

$$\approx \int_{\delta R}^{R} \left(\int_{\{r=s\}} |Df - U\hat{P}_v|^2 d\varphi \right) s\, ds$$

$$\left|\frac{\partial\varphi}{r}\right|_\delta = 1 \longleftarrow \quad \gtrsim \int_{\delta R}^{R} \left(\int_{\{r=s\}} \left| Df\left(\frac{\partial\varphi}{s}\right) - U\hat{P}_v\left(\frac{\partial\varphi}{s}\right) \right|^2 d\varphi \right) s\, ds$$

$$\text{Jensen} \longleftarrow \quad \geq \int_{\delta R}^{R} \frac{1}{2\pi s} \left| \int_{\{r=s\}} Df(\partial_\varphi) - U\tilde{P}_v(\partial_\varphi)\, d\varphi \right|^2 ds$$

$$= \int_{\delta R}^{R} \frac{1}{2\pi s} \left| \int_{\{r=s\}} Df - U\hat{P}_v \right|^2 ds$$

$$= |\underline{v}| \int_{\delta R}^{R} \frac{ds}{2\pi s} = \frac{|\underline{v}|^2}{2\pi} \log\frac{1}{\delta}$$

The admissible-strain model

$$E_M(f) = \int_M W(Df \circ P^{-1})\, d\mathrm{Vol}_{\bar{G}} \quad , \quad P = I + \frac{v}{2\pi} d\varphi$$

Assume:
- Small dislocation $P \sim I$
- small energy $E_M(f) << 1 \Rightarrow Df \approx U \in SO(2)$

Then: $U^T Df \circ P^{-1} \simeq U^T Df + (I-P) \equiv \beta$

Def: Admissible strains: $\{ \beta \in L^2(M;\mathbb{R}^{2\times 2}) \mid \operatorname{curl}\beta = 0 \,,\, \oint_\gamma \beta = -\underline{v} \}$

$$E^{as}(\beta) = \begin{cases} \int_M W(\beta)\, dx & \text{SZ12, MSZ14, ...} \\ \int_M Q(\beta - I)\, dx \quad , \quad Q(A) = \frac{1}{2} D_I^2 W(A,A) & \text{GLP10, ...} \end{cases}$$

Homogenization of dislocations

Single dislocation of size ε: $E \sim \varepsilon^2 \log \frac{1}{\varepsilon}$

n dislocations of size ε: $E_{n,\varepsilon} \sim \underbrace{n\varepsilon^2 \log \frac{1}{\varepsilon}}_{\text{self energy}} + \underbrace{\overset{\#\text{pairs}}{n^2}\varepsilon^2}_{\text{interaction energy}} \sim$ interaction between pair of disloc.

We would like to understand homogenization limits

$$\Gamma\text{-}\lim_{\substack{\varepsilon \to 0 \\ n_\varepsilon \to \infty}} \frac{1}{h_\varepsilon^2} E_{n,\varepsilon} \quad , \quad h_\varepsilon^2 = \max\{ n\varepsilon^2 \log \frac{1}{\varepsilon}, n^2\varepsilon^2 \}$$

Results:

- LGP'10, DGP'12,

  $\Gamma\text{-}\lim \frac{1}{h_\varepsilon^2} E^{as,lin}_{n,\varepsilon}$, $\log n_\varepsilon \ll \log \frac{1}{\varepsilon}$ $(h_\varepsilon \ll 1)$

- SZ'12, MSZ'14,'15 , Gin'19 , ...

  $\Gamma\text{-}\lim \frac{1}{h_\varepsilon^2} E^{lin}_{n,\varepsilon}$, $n_\varepsilon = \text{const}$ (SZ'12), $n_\varepsilon \sim \log \frac{1}{\varepsilon}$ (MSZ'14...)

- $\underset{\text{linear}}{\text{CGO'15}}$, $\underbrace{\text{GMS'21, CGM'22}}_{\text{non-linear}}$ 3D admissible strains, $\underbrace{\text{"}n_\varepsilon < C\text{"}}_{\text{line-tension models}}$

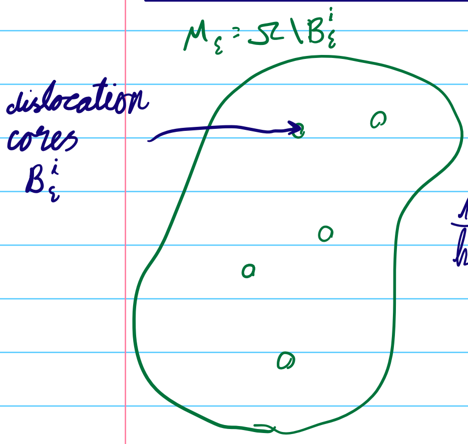
- KM'16, EKM'20

  $\Gamma\text{-}\lim E_{n,\varepsilon}$, $n_\varepsilon \sim \frac{1}{\varepsilon}$ $(h_\varepsilon \sim 1)$

- KM'23 Our focus today

  $\Gamma\text{-}\lim E_{n,\varepsilon}$, $\log n_\varepsilon \ll \log \frac{1}{\varepsilon}$ $(h_\varepsilon \ll 1)$

The limit in the admissible strain model (GLP'10, ...)



$$AS_\varepsilon = \{\beta\in L^2(M_\varepsilon;\mathbb{R}^{2\times2}) \mid \operatorname{curl}\beta=0,\ \oint_{\partial B^i_\varepsilon}\beta = -\varepsilon v^i_\varepsilon\}$$

$$\frac{1}{h_\varepsilon^2}E^{as}_\varepsilon(\beta_\varepsilon) = \frac{1}{h_\varepsilon^2}\int_{M_\varepsilon} Q(\beta_\varepsilon - I)\,dx$$

$$\Big\downarrow\ \Gamma \qquad \frac{1}{h_\varepsilon}(\beta_\varepsilon - I)\xrightarrow{L^2} J, \quad \underbrace{\frac{1}{n_\varepsilon}\sum v^i_\varepsilon\delta_{x^i_\varepsilon}}_{\text{"}\frac{1}{\varepsilon}\operatorname{curl}\beta\text{"}}\xrightarrow{*}\mu$$

$$E_0(\underbrace{J}_{\text{strain}},\underbrace{\mu}_{\text{dislocation density}}) = \underbrace{\int_\Omega Q(J)\,dx}_{\text{linear elastic energy}} + \underbrace{\int \Sigma\left(\frac{d\mu}{d|\mu|}\right)d|\mu|}_{\text{self-energy } (n_\varepsilon \lesssim \log\frac{1}{\varepsilon})}$$

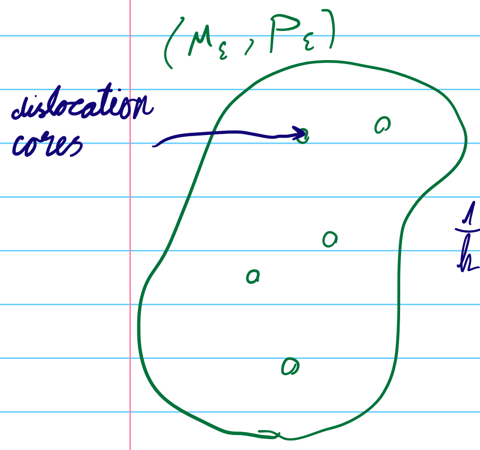
$$\operatorname{curl} J = \begin{cases} 0 & n_\varepsilon \ll \log\frac{1}{\varepsilon} \quad \text{elastic part is ind. of } \mu \\ -\mu & n_\varepsilon \gtrsim \log\frac{1}{\varepsilon} \end{cases}$$

Models of the type

$$E(u,\underbrace{\beta_p}_{\substack{\text{plastic strain}\\ \text{(additive decomposition)}}}) = \int Q(\underbrace{\nabla u - \beta_p}_{\text{elastic strain}}) + \int \varphi(\operatorname{curl}\beta_p)$$

are called "strain-gradient" models (Fleck & Hutchinson '93, Gurtin '00, ...)

The limit in the geometric model (KM'23)



$$f_\varepsilon \in H^1(M_\varepsilon; \mathbb{R}^2)$$

$$\frac{1}{h_\varepsilon^2} E_\varepsilon(f_\varepsilon) = \frac{1}{h_\varepsilon^2}\int_{M_\varepsilon} W(Df_\varepsilon \circ P_\varepsilon^{-1})\, dV_{G_\varepsilon}$$

$$\Big\downarrow \Gamma \qquad \begin{array}{ll} (M_\varepsilon, P_\varepsilon) \xrightarrow{n_\varepsilon} (\Omega, \mu) & ① \\ \frac{1}{h_\varepsilon}(R_\varepsilon^T Df_\varepsilon - P_\varepsilon) \xrightarrow{L^2} J & ② \end{array}$$

$$E_0(\underset{\text{strain}}{J}, \underset{\text{dislocation density}}{\mu}) = \underbrace{\int_\Omega \mathbb{Q}(J)\, dx}_{\text{linear elastic energy}} + \underbrace{\int \Sigma\left(\frac{d\mu}{d|\mu|}\right) d|\mu|}_{\text{self-energy } (n_\varepsilon \lesssim \log \frac{1}{\varepsilon})}$$

$$\operatorname{curl} J = \begin{cases} 0 & n_\varepsilon << \log\frac{1}{\varepsilon} \\ -\mu & n_\varepsilon \gtrsim \log \frac{1}{\varepsilon} \end{cases}$$

① **Manifold convergence** $(M_\varepsilon, P_\varepsilon) \to (\Omega, \mu)$ w.r.t. parameter n_ε if

$\exists Z_\varepsilon : M_\varepsilon \hookrightarrow \Omega$ uniformly bilipschitz s.t.

(i) $|\Omega \setminus M_\varepsilon| \to 0$ (ii) $\|P_\varepsilon - DZ_\varepsilon\|_{L^2} = O(h_\varepsilon)$

(iii) $|P_\varepsilon - DZ_\varepsilon| \lesssim \frac{\varepsilon}{\underbrace{r + o(1)}} \rightarrow$ distance to closest dislocation

(iv) $\frac{1}{n_\varepsilon \varepsilon}\int_{\partial M_\varepsilon} (\psi \circ Z_\varepsilon) \cdot P_\varepsilon \longrightarrow \int_\Omega \psi \cdot d\mu \quad \forall \psi \in C_c(\Omega)$ ("$\operatorname{curl} P_\varepsilon \xrightarrow{*} \mu$")

② **Convergence of strains**

Thm ("FJM"): If $(M_\varepsilon, P_\varepsilon) \xrightarrow{n_\varepsilon} (\Omega, \mu)$, then $\forall f_\varepsilon \in H^1(M_\varepsilon; \mathbb{R}^2)\ \exists R_\varepsilon \in SO(2)$

$$\int_{M_\varepsilon} |Df_\varepsilon - R_\varepsilon P_\varepsilon|^2 \leq C\left(\int_{M_\varepsilon} \operatorname{dist}^2(Df_\varepsilon \circ P_\varepsilon^{-1}, SO(2)) + h_\varepsilon^2\right)$$

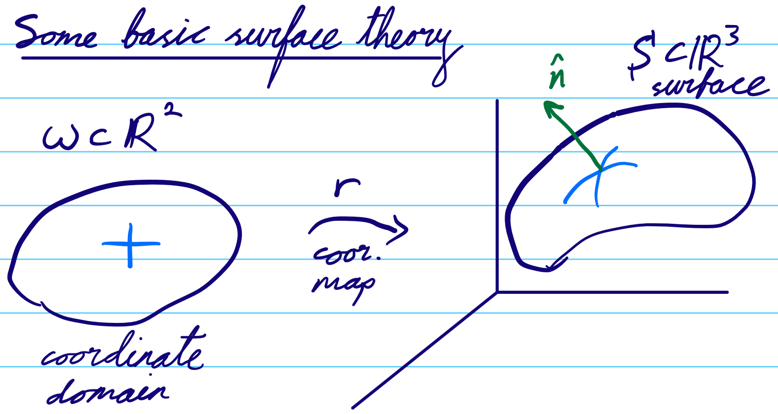
R_ε in ② is as in this thm.

Open questions

- Technical improvements in 2D:
  - Remove upper bounds on W (we assumed $W(A) \lesssim 1 + |A|^2$)
  - Remove separation assumptions
  - Compactness: - Manifold compactness
    - Supercritical case
  - Improved FJM?
  - Beyond $\log n_\varepsilon << \log \frac{1}{\varepsilon}$
- $O(1)$ energy limit w/o relaxation of core radius (improving EKM'20)
- 3D

III Thin non-Euclidean elastic bodies

Some basic surface theory



$\partial_i r \cdot \partial_j r \;=\; \bar{g}_{ij}$ (1^{st} fund. form - distance on S)

$-\partial_i r \cdot \partial_j n \;=\; \bar{\mathrm{I\!I}}_{ij}$ (2^{nd} fund. form - change of normal on S)

$\bar{g}$ & $\bar{\mathrm{I\!I}}$ satisfy the Gauss-Codazzi equations:

Gauss: $\bar{K} = \dfrac{\det \bar{\mathrm{I\!I}}}{\det \bar{g}}$, $\bar{K}$ = second order in $\bar{g}$

Codazzi: $\nabla_{[i}\mathrm{I\!I}_{j]k} = 0$, i.e. $\partial_2 \mathrm{I\!I}_{1k} - \partial_1 \mathrm{I\!I}_{2k} = \mathrm{I\!I}_{11}\Gamma^1_{k2} + \mathrm{I\!I}_{12}(\Gamma^2_{k2} - \Gamma^1_{1k}) - N\Gamma^2_{1k}$

$\bar{g}$ & $\bar{\mathrm{I\!I}}$ charecterize S up to rigid motion:

Thm: If $\bar{g}$ & $\bar{\mathrm{I\!I}}$ satisfy GC and ω is simply connected, then there exists a unique immersed surface (up to rigid motion) S with forms $\bar{g}, \bar{\mathrm{I\!I}}$.

Definition: $\bar{g}$ and $\bar{\mathrm{I\!I}}$ are symmetric $(2,0)$ tensors on TS.

The shape operator $\bar{S}$ is a $(1,1)$ tensor associated w. $\bar{\mathrm{I\!I}}$, i.e.

$\bar{g}(\bar{S}(v), w) = \bar{\mathrm{I\!I}}(v, w)$, or $\bar{g}_{ik} S^k_j = \mathrm{I\!I}_{ij}$.

It can be easily verified that $\bar{S}(v) = -\nabla_v \hat{n}$.

Skip

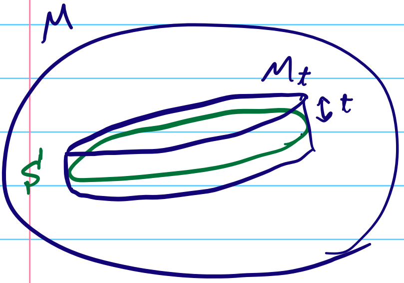
GC and Riemann's thm.

Remark: Define $\Omega = \omega \times (-\varepsilon_0, \varepsilon_0)$, and a metric $\bar G$ on Ω by

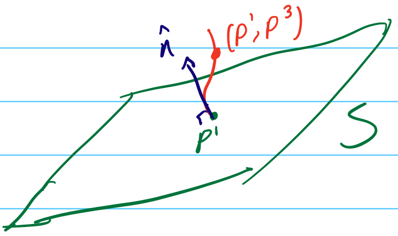
$$\bar G(x', x_3) = \left(\begin{array}{c|c} \bar g(x') - 2x_3 \bar{\mathrm{II}}(x') + x_3^2 \bar S^T_{(x')} \bar g(x') \bar S_{(x')} & \\ \hline & 1 \end{array}\right)$$

then $\bar g$ & $\bar{\mathrm{II}}$ satisfy GC iff $\bar G$ is flat, i.e. if $(\Omega, \bar G)$ can be (locally) isometrically-immersed in $\mathbb{R}^3$.

In the other direction, given a 3-dim. manifold M, and a subman. S



the t-tubular neigh. of S, M_t, is obtained by going via normal geodesics up to time t:



The metric $\bar G$, restricted to $\bar M_t$, satisfies the following expansion:

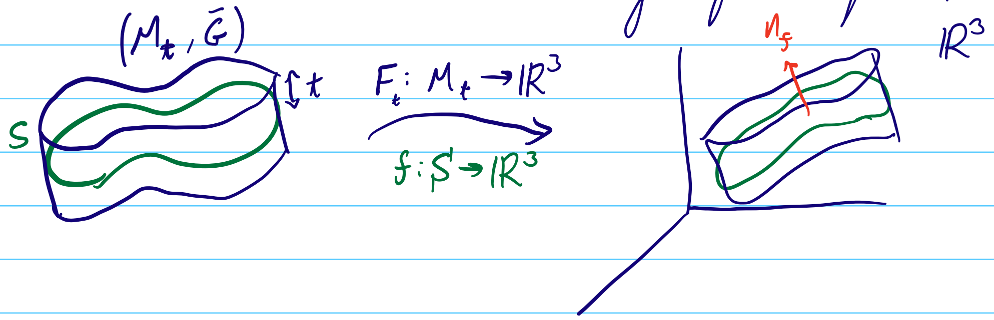
$$\bar G(p', p^3) = \left(\begin{array}{c|c} \bar g(p') - 2p^3 \bar{\mathrm{II}}(p') + O(|p^3|^2) & \\ \hline & 1 \end{array}\right)$$

where $\bar g$ is the induced metric on S, and $\bar{\mathrm{II}} = \mathrm{II}_{S,M}$ is the second form of S in M.

$\bar g$ & $\bar{\mathrm{II}}$ satisfy Gauss-Codazzi iff $R^{\bar G}(X,Y) = 0 \ \forall X, Y \in TS$, i.e., if $R^{\bar G}_{12ij}\big|_S \equiv 0$.

Elastic energy - formal asymptotics

M_t – thin 3-dim. man., t-tub. neigh. of a surface S



Assume $F_t(p', p^3) = f(p') + p^3 n_f(p')$, and that $W(A) = \mathrm{dist}^2(A, SO(3))$

Then, formally,

$$E_{M_t}(F_t) = \underbrace{\underbrace{\int_S \mathrm{dist}^2\Big(Df \circ \bar{g}^{-\frac{1}{2}}, \overset{\{A:\mathbb{R}^2\to\mathbb{R}^3 \,|\, A^TA = I_{2\times 2}\}}{\overset{\shortparallel}{\overbrace{O(2,3)}}}\Big)\, dV_{\bar{g}}}_{\text{stretching } \; E_S^{str}(f)} + \underbrace{\frac{t^2}{3}\int_S |\nabla n_f + Df \circ \bar{S}|^2 \, dV_{\bar{g}}}_{\text{bending } \; t^2 E_S^{bnd}(f)}}_{E_S(f) \qquad \text{Kirchhoff shell energy}} + O(t^3)$$

Remark: An alternative bending energy, which differs from the above by higher order terms, is

$$\frac{t^2}{3}\int_S |\mathrm{II}_f - \bar{\mathrm{II}}|^2 \, dV_{\bar{g}}$$

where $\mathrm{II}_f = Df^T \nabla n_f$.
While more common in the physics community, it is worse from analytic point of view, because of the product structure of II_f.

Note that

$$\{f : S \to \mathbb{R}^3 \,|\, E_S(f) < \infty\} = \{f \in W^{1,2}(S; \mathbb{R}^3) \,|\, \mathrm{rnk}\, Df = 2 \text{ a.e.} \overset{\text{implies existence}}{\curvearrowright} , n_f \in W^{1,2}(S; \mathbb{R}^3)\} =: \mathrm{Imm}^2(S; \mathbb{R}^3)$$

Interesting "critical" Sobolev space, similar to $\{f \in W^{1,2}(S; \mathbb{R}^2) \,|\, \det Df > 0\}$

Open question: $\mathrm{Imm}^2(S; \mathbb{R}^3) \subset C(S; \mathbb{R}^3)$? (This is true if $\nabla n_f = 0$ Goldstein, Šverak, ...)

Does Kirchhoff energy allows for cavitation?

Thm (AKM'22): $\inf E_S = 0 \Leftrightarrow \exists f \in C^\infty(S, \mathbb{R}^3)$ iso. imm w. forms $\bar{g}, \bar{I\!I}$.
In particular, $\bar{g}$ & $\bar{I\!I}$ satisfy GC.
If S is simply-connected, this also implies $\inf E_S = 0$.

Skip

Pf (sketch): "undo the dimension reduction".

Define a manifold $M = S^1 \times (-\varepsilon_0, \varepsilon_0)$ w. metric

$$\bar{G}_{(x'; x_3)}((v,s),(w,r)) = \bar{g}_{x'}(v - s\bar{S}(v), w - r\bar{S}(w)) + sr.$$

This only works for const. curv. target space. For general targets a direct approach is needed (AKM'23)

Given $f \in \mathrm{Imm}^2(S^1; \mathbb{R}^3)$, define $F \in W^{1,2}(M; \mathbb{R}^3)$ by

$$F(x', x^3) = f(x') + x^3 n_f(x')$$

Then $\int_M \mathrm{dist}^2(DF \circ \bar{G}^{-\frac{1}{2}}, SO(3)) \leq C E_S(f)$.

Thus, $\inf E_S = 0$ implies $\inf E_M = 0$, and thus $\exists$ smooth iso. imm. $F: M \to \mathbb{R}^3$. $F|_{S^1 \times \{0\}}$ is the wanted map. ▨

Non-Euclidean shells – rigorous limits

As $t \to 0$ we can see that stretching becomes infinitely costly compared to bending, thus we expect

$$\lim_{t \to 0} \frac{1}{t^2} E_{M_t} = \begin{cases} E_S^{bnd} & E_S^{str} = 0 \\ +\infty & E_S^{str} \neq 0 \end{cases}$$

Note that if $E_S^{str}(f) = 0$ then $Df^T Df = \bar{g}$ a.e., and thus

1) $f \in Imm^2(S; \mathbb{R}^3) \Rightarrow f \in W^{2,2}(S; \mathbb{R}^3)$, since $\partial_{ij} f = \bar{\Gamma}^k_{ij} \partial_k f - \bar{\Gamma}^k_{il} \bar{g}_{kj} \bar{g}^{lm} n_m$

   We denote this space by $W^{2,2}_{\bar{g}}(S; \mathbb{R}^3)$.

2) $E_S^{bnd}(f) = \frac{1}{3} \int_S |\mathrm{II}_f - \bar{\mathrm{II}}|^2 dV_{\bar{g}}$

<u>Thm</u> (Sch'07, LP'11, BLS'16, KS'14): $\leftarrow$ generalizations of FJM'02, FJMM'03

- $E_{M_t}(f_t) \leq C t^2 \Rightarrow f_t \xrightarrow{W^{1,2}} f \in W^{2,2}_{\bar{g}}(S; \mathbb{R}^3)$ (after rescaling ...)
- $\frac{1}{t^2} E_{M_t} \xrightarrow{\Gamma} \begin{cases} E_S^{bnd}(f) & f \in W^{2,2}_{\bar{g}} \\ +\infty & \text{else} \end{cases}$

Corollary: (MS'19)

- $\inf E_{\mathcal{M}_t} = O(t^2) \iff W^{2,2}_{\bar{g}}(S;\mathbb{R}^3) \neq \phi$
- $\inf E_{\mathcal{M}_t} = o(t^2) \iff \exists f \in W^{2,2}_{\bar{g}}(S;\mathbb{R}^3) \cap C^\infty, \mathrm{II}_f = \bar{\mathrm{II}} \underset{\overset{\Leftarrow}{S \text{ simply-connected}}}{\Rightarrow} \bar{g}, \bar{\mathrm{II}}$ satisfy GC

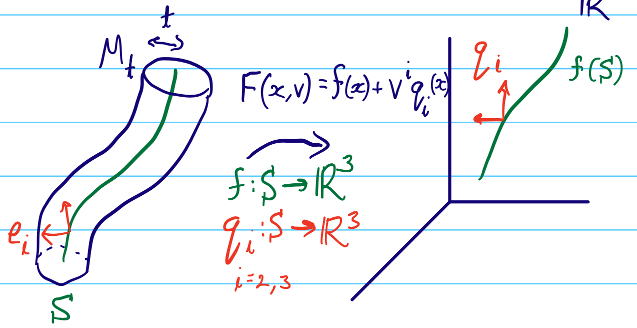
  $\iff \inf E_{\mathcal{M}_t} = O(t^4)$
- $\inf E_{\mathcal{M}_t} = o(t^4) \underset{\overset{\Leftarrow}{\text{simply conn.}}}{\Rightarrow} \mathcal{R}^{\bar{G}}|_S \equiv 0$ (follows from local estimate)

Higher order scaling in Lewicka '20.

If time permits

Rods

For rods we have a different behavior



Thm (KS '14, Aharoni et al. '12):

- $E_{\mathcal{M}_t}(f_t) \le C t^2 \Rightarrow f_t \xrightarrow{W^{1,2}} f \in W^{2,2}(S;\mathbb{R}^3), \quad Df_t \xrightarrow{L^2} \overbrace{(Df|q_2|q_3)}^{q} \in W^{1,2}(S;SO(\bar{G}))$
- $\frac{1}{t^2} E_{\mathcal{M}_t} \xrightarrow{\Gamma} \begin{cases} C\int |q_2' \cdot q_1 - \underbrace{\bar{\Gamma}^1_{12}}_{\kappa_1}|^2 + |q_3' \cdot q_1 - \underbrace{\bar{\Gamma}^1_{13}}_{\kappa_2}|^2 + |q_3' \cdot q_2 - \underbrace{\bar{\Gamma}^2_{13}}_{\tau}|^2 & q \in SO(\bar{G}) \\ +\infty \end{cases}$

For $W(\cdot) = \mathrm{dist}^2(\cdot, SO(3))$, round cross-section. Otherwise an equivalent energy

Since we can always solve $q' = \begin{pmatrix} 0 & -\bar{\Gamma}^1_{12} & -\bar{\Gamma}^1_{13} \\ \bar{\Gamma}^1_{12} & 0 & -\bar{\Gamma}^2_{13} \\ \bar{\Gamma}^1_{13} & \bar{\Gamma}^2_{13} & 0 \end{pmatrix} q$ (no compatibility conditions)

Cor (MS'19):
- $\inf E_{\mathcal{M}_t} = O(t^4)$
- $\inf E_{\mathcal{M}_t} = o(t^4) \Rightarrow \mathcal{R}^{\bar{G}}|_S \equiv 0.$

Summary:	Shells	Rods
Generic scaling of E_{M_t}:	t^2	t^4
Behavior of min:	$\min \int \lvert \mathrm{II}_f - \bar{\mathrm{II}} \rvert^2$ II_f subject to GC	" $\mathrm{II}_f = \bar{\mathrm{II}}$ " (satisfy curvature, torsion)

If time permits

Comment: In KS'14 dimension reduction at the t^2 scaling is done

$$\text{for any } \underbrace{\dim S}_{m} < \underbrace{\dim M}_{n} \hookrightarrow \mathbb{R}^n \qquad k = n - m$$

We can choose a orthonormal frame $e_1, \dots, e_k$ of NS, which induce coordinates on M s.t.

$\langle \nabla^{\bar{G}}_{\partial_i} \partial_j , e_\alpha \rangle = (\mathrm{II}_\alpha)_{ij}$ second forms (sym. mat.)

$\langle \nabla^{\bar{G}}_{\partial_i} e_\alpha , e_\beta \rangle = (\tau_i)_{\alpha\beta}$ is the twist matrices (skew)

Then we have

- $E_{M_t}(f_t) \le C t^2 \Rightarrow f_t \xrightarrow{W^{1,2}} f \in W^{2,2}_{\bar{g}}(S; \mathbb{R}^n)$, $Df \xrightarrow{L^2} (Df \,|\, \overset{q_1^\perp, \dots, q_k^\perp}{\overset{"}{q^\perp}}) \in W^{1,2}(S, SO(\bar{G}))$

- $\Gamma\text{-}\lim \frac{1}{t^2} E_{M_t} \overset{\text{if not } +\infty}{=} C(m,k) \int_S \sum_\alpha 2 \lvert \underbrace{P^{\parallel}_F}_{\text{proj. on } T(F(S))} \circ \nabla_{\cdot} q^\perp_\alpha + DF \circ \underbrace{S^{(\cdot)}_\alpha}_{\text{shape operator associated w. } \mathrm{II}_\alpha} \rvert^2 + \lvert P^\perp_F \circ \nabla_{\cdot} q^\perp_\alpha - \sum_\beta (\tau_\cdot)_{\alpha\beta} q^\perp_\beta \rvert^2$

" " $N(F(S))$

for $W(\cdot) = \mathrm{dist}^2(\cdot, SO(n))$
and circular cross-section

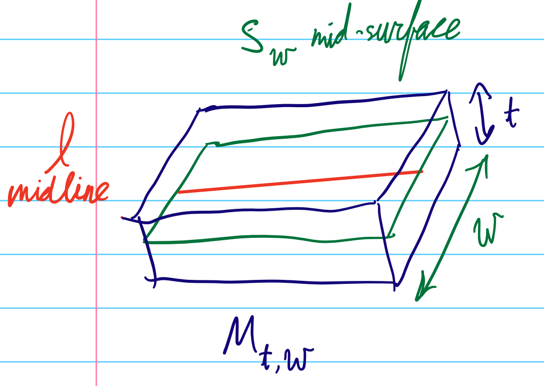
Ribbons

ribbons interpolate between these behaviors

Summary:

	Shells ⟷	Rods
Generic scaling of E_{M_t}:	t^2	t^4
Behavior of min:	$\min \int \lvert \mathrm{II}_f - \bar{\mathrm{II}} \rvert^2$ II_f subject to GC	"$\mathrm{II}_f = \bar{\mathrm{II}}$" (satisfy curvature, torsion)

↓ Ribbons see the difference between Gauss & Codazzi



$t \ll w \ll 1$

$\boxed{\text{Show examples}}$

$$E_{t,w}(f) = \int_{M_{t,w}} \mathrm{dist}^2(Df \circ \bar{G}^{-1/2}, SO(3))\, dV_{\bar{G}}$$

$$E^{2D}_{t,w}(f) = \int_{S_w} \mathrm{dist}^2(Df \circ \bar{g}^{-1/2}, SO(2,3))\, dV_{\bar{g}} + t^2 \int_{S_w} \lvert Dn_f + Df \circ \bar{S} \rvert^2 dV_{\bar{g}}$$

$$\approx \int_{S_w} \lvert \underbrace{Df^T Df}_{g_f} - \bar{g} \rvert^2 dV_{\bar{g}} + t^2 \int_{S_w} \lvert \underset{\underset{-Df^T Dn_f}{\downarrow}}{\mathrm{II}_f} - \bar{\mathrm{II}} \rvert^2 dV_{\bar{g}}$$

Example 1: $S^w = (0,\ell) \times (-\frac{w}{2}, \frac{w}{2})$, $\bar{g} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\bar{\mathrm{II}} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$

✗ **Gauss:** $\det \bar{\mathrm{II}} = -\lambda^2$, $K_{\bar{g}} \det \bar{g} = 0$

✓ **Codazzi:** $\partial_i \mathrm{II}_{jk} = 0 = \partial_j \mathrm{II}_{ik}$ → since $\nabla_i^{\bar{g}} = \partial_i$

$$\bar{g}_f = \begin{pmatrix} 1 - K_0(x_1) x_2 + O(x_2^2) & 0 \\ 0 & 1 \end{pmatrix}$$

→ Gaussian curvature of $f(S_w)$ along the midline.

Thus, formally,

$$E^{2D}_{t,w} = w^4 \int_L K_0^2(x_1)\, dx_1 + t^2 \int_L |\mathrm{II}_f(x_1,0) - \bar{\mathrm{II}}|^2 dx_1 + h.o.t$$

leading order constraint: $K_0(x_1) = \det \mathrm{II}_f(x_1,0)$

Cor: Wide ribbon $(t << w^2)$ $\underset{\text{dev. surface}}{K_0 \equiv 0}$ $E^{2D}_{t,w} \sim t^2 \int_L |\underbrace{\mathrm{II}_f(x_1,0)}_{\det \mathrm{II}_f = 0} - \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}|^2 dx_1 \sim t^2 \lambda^2$

Narrow ribbon $(w^2 << t)$ $\underset{\text{pure twist}}{\mathrm{II}_f(x_1,0) = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}}$ $\Rightarrow K_0 = -\lambda^2$ $E^{2D}_{t,w} \sim w^4 \lambda^4$

$\Rightarrow$ transition @ $t \sim w^2$

Example 2: $S^w = (0,l) \times (-\frac{w}{2}, \frac{w}{2})$, $\bar{g} = \begin{pmatrix} (1-\kappa x_2)^2 & 0 \\ 0 & 1 \end{pmatrix}$, $\bar{\mathrm{II}} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$

✓ Gauss: $\det \bar{\mathrm{II}} = 0 = K_{\bar{g}}$

× Codazzi: $\nabla_2 \bar{\mathrm{II}}_{11} - \nabla_1 \bar{\mathrm{II}}_{21} = -\lambda\kappa(1-\kappa x_2)^2 \neq 0$

$K_0(x_1) = K_{g_f}(x_1,0)$, $K_1(x_1) = \partial_2 K_{g_f}(x_1,0)$

It is always energetically favorable to respect the reference geodesic curvature of the mid-line

$$g_f = \begin{pmatrix} (1-\kappa x_2)^2 - K_0 x_2^2 + \frac{1}{3}(4\kappa K_0 - K_1) x_2^3 + \dots & 0 \\ 0 & 1 \end{pmatrix}$$

having $\mathrm{II}_f(x_1,0) = \bar{\mathrm{II}}$ is consistent with $K_0 = \bar{K} = 0$, but then

by Codazzi we get $\partial_2 \mathrm{II}_f(x_1,0) = -\lambda\kappa$ hence $\mathrm{II}_f = \begin{pmatrix} -\lambda\kappa x_2 & 0 \\ 0 & \lambda \end{pmatrix} + h.o.t.$

hence if $\mathrm{II}_f(x_1,0) = \bar{\mathrm{II}}$ we get $K_1 = -\lambda^2\kappa \neq 0$, i.e. effectively we have

$$E^{2D}_{t,w} = \underbrace{w^6 \int_L K_1^2(x_1)\, dx_1 + t^2 \int_L |\mathrm{II}_f(x_1,0) - \bar{\mathrm{II}}|^2 dx_1}_{\text{cannot be simultaneously zero}} + h.o.t.$$

Thus we expect transition between $\mathrm{II}_f = \bar{\mathrm{II}}$ for narrow ribbons $t >> w^3$ and $K_f = O(x_2^2)$ for wide ribbons $t << w^3$.
However, it can be shown that the transition can be moved at least to $t \sim w^{\frac{8}{3}}$

Non-Euclidean ribbons

(Levin-Siéfert-Sharon-M., 2021)

Gauss incom.

$\bar{\mathrm{II}}$ (preferred bending)

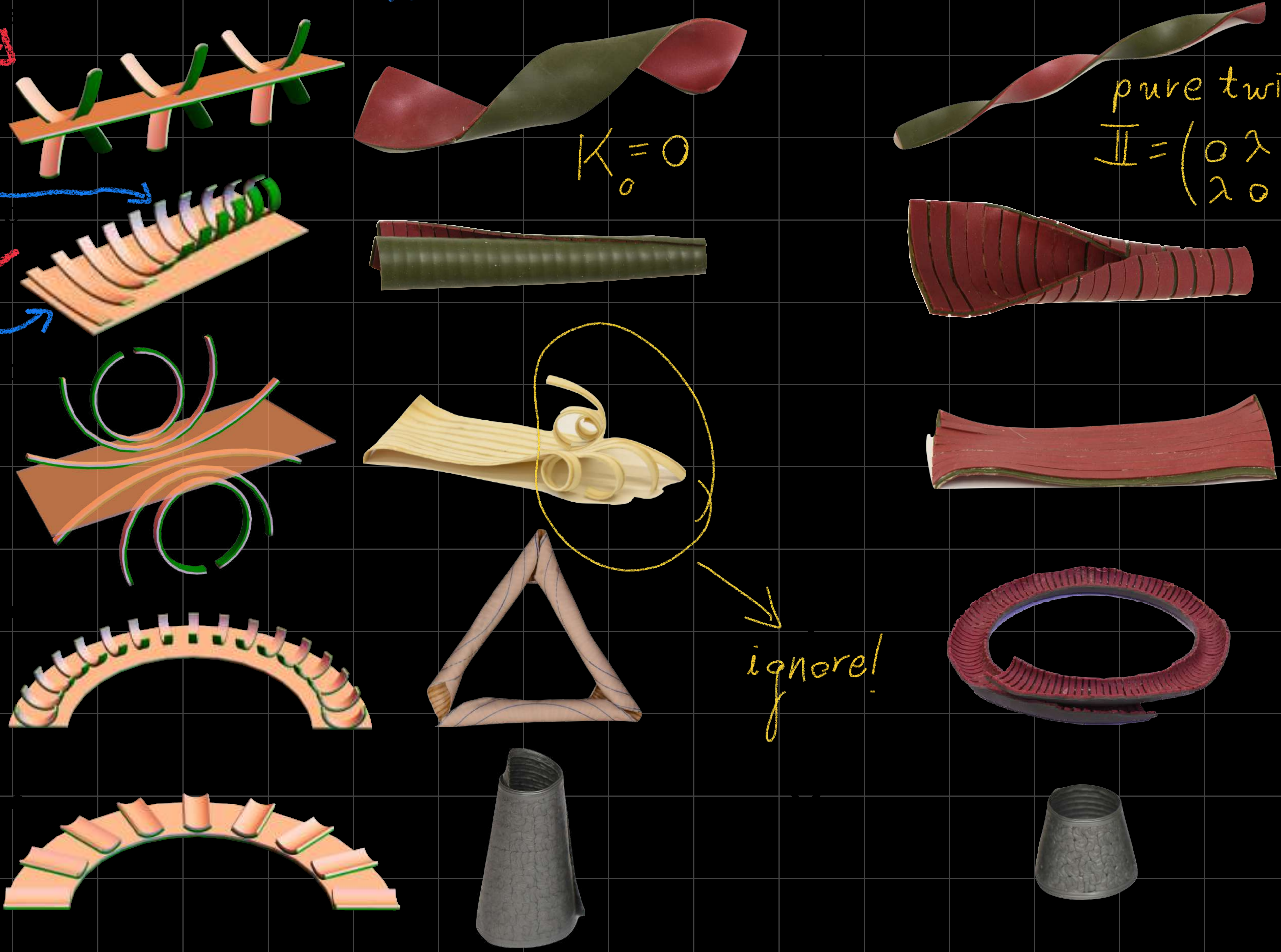
$\bar{g}$ (flat)

Codazzi incomp.

wide ribbon

narrow ribbon

shape transition



$K_0 = 0$

pure twist

$\mathrm{II} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$

ignore!

✓ $t \sim w^2$

✓ $t \sim w^4$

×

✓ $t \sim w^3$

×