Slowdown estimates for random walk in random environment

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Definition

Fix $d \geq 1$.

Let $\mathcal{M}^d$ denote the space of all probability measures on $\mathcal{E}_d = \{0\} \cup \{\pm e_i\}_{i=1}^d$

Let $\Omega = \left(\mathcal{M}^d\right)^{\mathbb{Z}^d}$.

An *environment* is a point $\omega = \{\omega(x,e)\}_{x \in \mathbb{Z}^d,\, e \in \mathcal{E}_d} \in \Omega$.

Let P be a translation invariant (ergodic) probability measure on Ω.

Definition

For $\omega \in \Omega$ and $z \in \mathbb{Z}^d$ define:

P_ω^z is the distribution of a Markov process $\{X_n\}$ with

$$X_0 = z$$

and

$$P_\omega^z(X_{n+1} = x + e | X_n = x) = \omega_x(e)$$

for all $e \in \mathcal{E}_d$.

Notation

P_ω^z is called the **quenched** law

$$\mathbb{P} = P \otimes P_\omega^z$$

Is the joint distribution of the environment and the walk.

$$\mathbf{P}^z(\cdot) = \int_\Omega P_\omega^z(\cdot) dP(\omega)$$

is the **annealed** law.

If $z = 0$ we omit the superscript.

Assumptions

We make the following assumptions:

1. The environment is i.i.d, namely $P = Q^{\mathbb{Z}^d}$ for some Q.
2. The walk is “uniformly elliptic”, i.e. there exists $\kappa > 0$ such that Q-almost surely for every e,

$$\omega(e) > \kappa.$$

Example : Arrow model

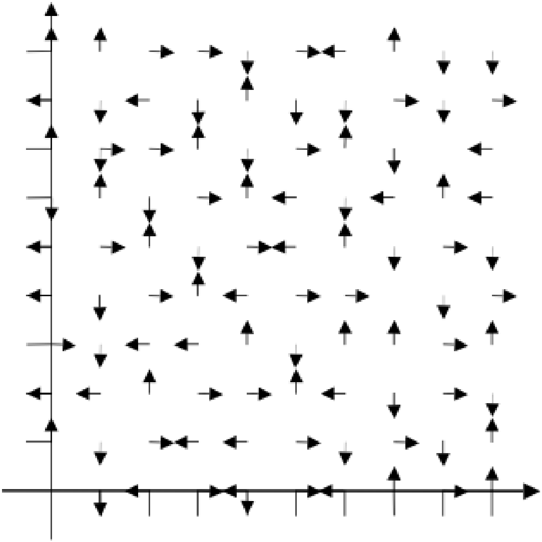
Fix $0 < \epsilon < 1$.

Let $\eta : \mathbb{Z}^d \to \mathcal{E}_d$ be i.i.d. uniform.

We take

$$\omega_z(e) = \begin{cases} \epsilon & \text{if } e = \eta(z) \\ \frac{1-\epsilon}{2d-1} & \text{otherwise} \end{cases} .$$

Arrow model



Definition

We say that the system is *ballistic* if there exists $v \neq 0$ in $\mathbb{R}^d$ such that

$$\mathbf{P}\left(\lim_{n\to 0} \frac{X_n}{n} = v\right) = 1.$$

There is no known effective characterization of ballisticity.

Question

For $a \neq v$ and large n, what is the probability that

$$X_n \approx na?$$

This is a *large deviation* type of question.

Nestling

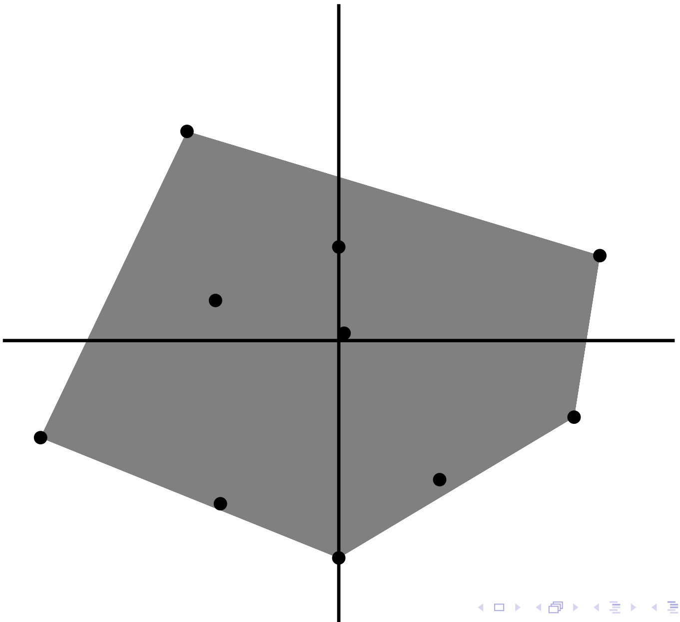
The *local drift* at z is defined to be

$$E_\omega^z(X_1) - z.$$

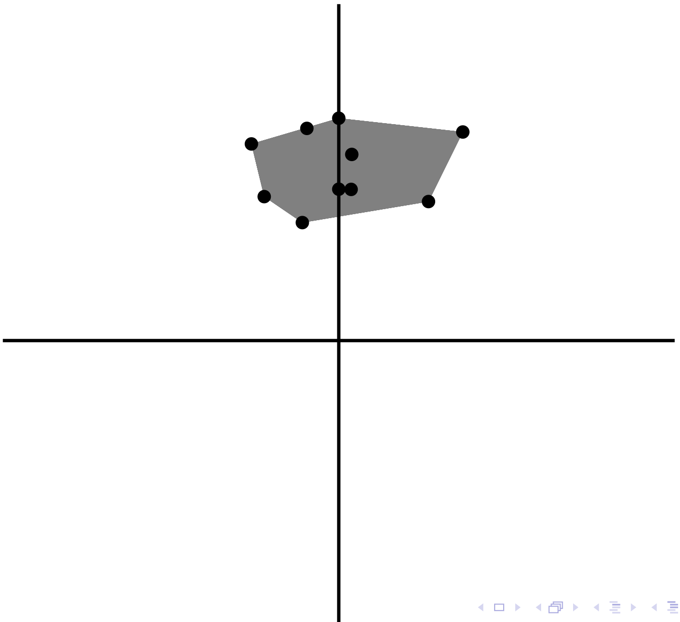
We say that the system is *nestling* if 0 is in the convex hull of the support of the local drift,

and that it is *non-nestling* otherwise.

Nestling



Non-nestling



Varadhan's Theorem for the non-nestling case

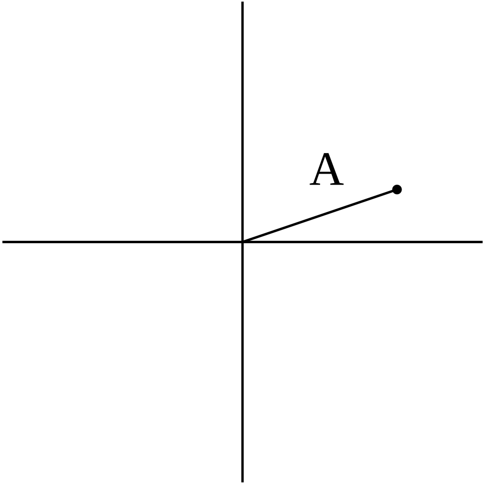
There exists a convex function $F : \mathbb{R}^d \to \mathbb{R}^+$, such that $F(v) = 0$ and $F > 0$ outside v, such that

$$\mathbf{P}(X_n \approx an) \approx e^{nF(a)}.$$

i.e. for every $a \neq v$, the decay is exponential.

Varadhan's Theorem for the nestling case

Let A be the line connecting the origin to v.



Varadhan's Theorem for the nestling case

Let A be the line connecting the origin to v.

Then, $F^{-1}(0) = A$.

In other words,
the probability of slowdown of the walk decays slower than exponentially.

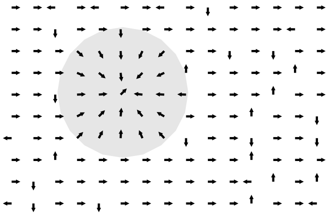
Question: What is the rate of the decay of the probability of slowdown?

Lower bound

For every $a \in A$ there exists C such that

$$\mathbf{P}(X_n \approx an) > e^{-C(\log n)^d}.$$

Lower bound - proof

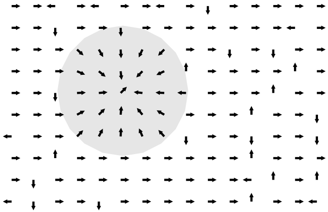


Assume that the "trap" is of radius $\alpha \log n$, with α being a large constant.

With high probability, the trap holds the walker for (at least) a linear amount of time.

The probability of existence of such a trap is exponential in its volume, $(\log n)^d$.

So, the probability of a linear slowdown is at least $\exp(-C(\log n)^d)$.



Is this the way slowdown occurs?

Sznitman's condition (T)

The following condition, named condition (T), is conjectured to be equivalent to ballisticity.

Notation: For $\ell \in S^{d-1}$ and $L \in \mathbb{R}^+$, we define

$$T_L^{(\ell)} := \min\{n : \langle X_n, \ell \rangle > L.$$

Condition: There exist a non-empty open set of directions, $G \in S^{d-1}$, such that for every $\ell \in G$ there exists $\gamma > 0$ such that for all large L

$$\mathbf{P}\left(T_L^{(\ell)} > T_L^{(-\ell)}\right) < e^{-\gamma L}.$$

Known upper bound

Assume Condition (T), and $d \geq 2$.

For every $a \in A$ and $\alpha = \frac{2d}{d+1}$, if n is large enough, then

$$\mathbf{P}(X_n \approx an) < e^{-(\log n)^{\alpha}}.$$

Sztitman 2001.

Main result

Assume Condition (T), and $d \geq 4$.

For every $a \in A$ and every $\epsilon > 0$, if n is large enough, then

$$\mathbf{P}(X_n \approx an) < e^{-(\log n)^{d-\epsilon}}.$$

Regeneration times

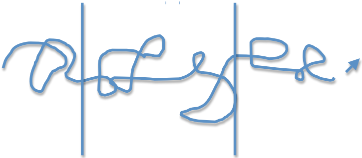


Figure: Regeneration

t is said to be a *regeneration time* if:

1. $\langle X_s, \ell \rangle < \langle X_t, \ell \rangle$ for all $s < t$.
2. $\langle X_s, \ell \rangle > \langle X_t, \ell \rangle$ for all $s > t$.

Regeneration times

Facts (Sznitman + Zerner 2000):

1. Almost surely, there are infinitely many regeneration times. we call them $\tau_1 < \tau_2 < \ldots$
2. The ensemble

$$\left\{(\tau_{n+1} - \tau_n), (X_{\tau_{n+1}} - X_{\tau_n})\right\}_{n=1}^{\infty}$$

is an i.i.d. ensemble.

Proposition

For all $\epsilon > 0$ and u large enough,

$$\mathbf{P}(\tau_1 > u) \leq e^{-(\log u)^{d-\epsilon}}.$$

Proof of main result assuming proposition

Let

$$\rho = \mathbf{E}(\tau_2 - \tau_1)$$

and

$$\alpha = \mathbf{E}\left(\langle X_{\tau_2} - X_{\tau_1}, e_1\rangle\right).$$

Let

$$\eta = \frac{\alpha}{\rho},$$

let $b = a/v$ and let $m = \left[n \cdot \frac{1+b}{2} \cdot \frac{1}{\rho}\right]$.

Proof of main result assuming proposition

Then,

$$\mathbf{P}(X_n \approx an) \leq \mathbf{P}(\tau_m > n) + \mathbf{P}(\langle X_{\tau_m}, e_1 \rangle < b\alpha).$$

By condition (T),

$$\mathbf{P}(\langle X_{\tau_m}, e_1 \rangle < b\alpha)$$

decays exponentially,

and thus we need to control

$$\mathbf{P}(\tau_m > n).$$

Proof of main result assuming proposition

By the proposition, for every k,

$$\mathbf{P}(\tau_k - \tau_{k-1} > n^{1/8}) \leq \frac{1}{2n} e^{-(\log n)^{d-\epsilon}},$$

and by Azuma's inequlity

$$\mathbf{P}(\tau_m > n \,|\, \forall_{k \leq m} \tau_k - \tau_{k-1} \leq n^{1/8}) \leq e^{-n^{1/2}}.$$

Therefore, all we need to do is to prove the proposition,

namely, that for all $\epsilon > 0$ and u large enough,

$$\mathbf{P}(\tau_1 > u) \leq e^{-(\log u)^{d-\epsilon}}.$$

Reduction

Let $L = (\log u)^d$.

Using condition (T),

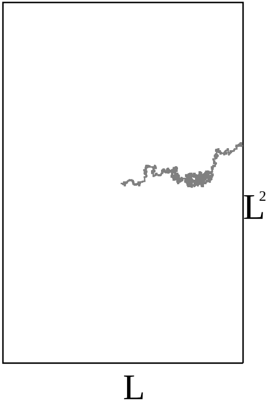
$$\mathbf{P}(\tau_1 > u) \leq \mathbf{P}(T_L > u) + e^{-O\left((\log u)^d\right)}$$

Thus all we need is to estimate $\mathbf{P}(T_L > u)$.

This enables us to estimate the amount of time to a stopping time.

Reduction

Let B_L be the box of side-length $2L$ and width L^2 around the origin.



Reduction

Now,

$$\mathbf{P}(T_L > u) \leq \mathbf{P}(T_{B_L} > u) + e^{-O\left((\log u)^d\right)}$$

and

$$\mathbf{P}(T_{B_L} > u) \leq \mathbf{P}(\exists_{x \in B_L} \text{ such that } x \text{ is visited } \frac{u}{|B_L|} \text{ times before } T_{B_L}).$$

So all we need is to bound

$$\mathbf{P}(\exists_{x \in B_L} \text{ such that } x \text{ is visited } \frac{u}{|B_L|} \text{ times before } T_{B_L}).$$

Reduction

For every x and every event $G \subseteq \Omega$ on the environments,

$$\begin{aligned} &\mathbf{P}\left(x \text{ is visited } \frac{u}{|B_L|} \text{ times before } T_{B_L}\right) \\ &\leq P(G^c) + \sup_{\omega \in G} P_\omega\left(x \text{ is visited } \frac{u}{|B_L|} \text{ times before } T_{B_L}\right). \end{aligned}$$

and by the Markov property,

$$\begin{aligned} &P_\omega\left(x \text{ is visited } \frac{u}{|B_L|} \text{ times before } T_{B_L}\right) \\ &\leq P_\omega^x\left(x \text{ is visited } \frac{u}{|B_L|} \text{ times before } T_{B_L}\right) \\ &= \left(P_\omega^x(\text{ return to } x \text{ before } T_{B_L})\right)^{\frac{u}{|B_L|}}. \end{aligned}$$

Reduction

Therefore, we need to find an event $G \subseteq \Omega$ such that

1. $P(G) > 1 - e^{-(\log u)^{d-\epsilon}}$.

2. For every $\omega \in G$,

$$1 - P_\omega^x(\text{ return to } x \text{ before } T_{B_L}) >> \frac{1}{u}.$$

The event G

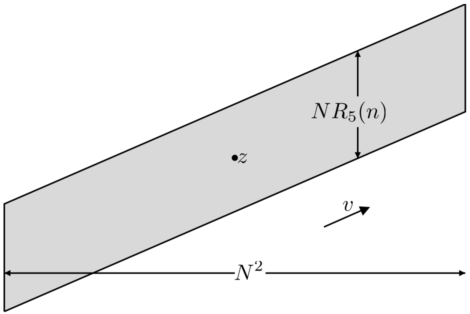
For $n > 0$, let $A_n \subseteq \Omega$ be the following event:

1. $P_\omega(T_{-n} < T_n) < e^{-cn}$.
2. The quenched distribution of X_{T_n} is very closed to the annealed in the following sense: There exists a coupling between the two distributions, such that with probability λ their distance is less than n^ϵ, and $\lambda = \lambda(n)$ is very small.

Lemma: $1 - P(A_n)$ decays faster than any polynomial.

The event G

For every n, partition the lattice into parallelograms in the direction of the speed, of length n^2 and width a little more than n.



We can now define the event G.

The event G

We say that a parallelogram of length n^2 is **good** if the event A_n holds for the walk starting from its center.

Note that these events are almost independent for disjoint blocks.

Now, let $n_1 = L^{\epsilon}, n_2 = L^{2\epsilon}, \ldots$.

The event G is the event that in every such scale, the number of bad parallelograms in B_L is no more than $(\log u)^{d-\epsilon}$.

It is easy to see that $P(G) > 1 - e^{-\log(u)^{d-\epsilon}}$. Therefore all we need to show is that for every $\omega \in G$,

$$1 - P_{\omega}^{x}(\text{ return to } x \text{ before } T_{B_L}) >> \frac{1}{u}.$$

The quenched escape probability

We need to show that for $\omega \in G$,

$$1 - P_\omega^x(\text{ return to } x \text{ before } T_{B_L}) >> \frac{1}{u}.$$

To see this we define an event A, and show that

1. $P_\omega^x(A) >> \frac{1}{u}$, and
2. On the event A, the walker leaves B_L before returning to x.

The quenched escape probability

We first define an event B as follows:

The event B is the event that for every parallelogram that the walker visits, it exits through the front, and that whenever it goes through a bad parallelogram, at the exit it “corrects” its position to be similar to the annealed. The correction is done using ϵ-coins.

Conditioned on the event B, the walker does not return to x, and its path looks like Brownian Motion.

The quenched escape probability

We now define the event A as follows:

Let w be a random variable, uniform in the set $[-1,1]^{d-1}$ and independent of the walk.

The event A is the following event:

$$A = B \cap \left\{ \forall_k, X_{T_{J_k}} - X_{T_{J_{k-1}}} - e_1(J_k - J_{k-1}) - w(J_k - J_{k-1})n_k < n_k \right\}$$

where $J_1 = n_1(\log u)^{d-\epsilon}$ and $J_k = J_{k-1} + n_k(\log u)^{d-\epsilon}$.

The quenched escape probability

Conditioned on the event A, with high probability the walks visit no more than $(\log u)^{1-\epsilon}$ bad blocks.

Therefore, under this event it needs no more than $(\log u)^{1-\epsilon}$ ϵ-coins.

$$P(A|B) > u^{\epsilon-1}.$$

Combined, we get that

$$P\omega(A) >> \frac{1}{u}.$$

□

THANK YOU