Quenched central limit theorem for random walk on percolation clusters

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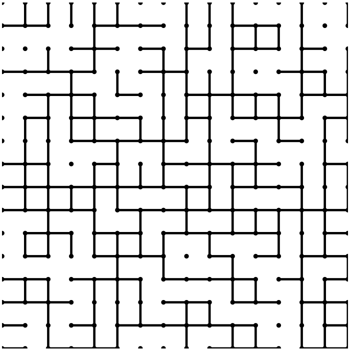
Bond-percolation in $\mathbb{Z}^d$

Fix some parameter $0 < p < 1$, and for every edge e in $\mathbb{Z}^d$, independently of all other edges, declare that e is "open" with probability p and "closed" with probability $1 - p$. We are interested in the (random) graph spanned by the vertices of $\mathbb{Z}^d$ and the open edges.



Simple random walk on the infinite cluster

For $p > p_c(d)$, with positive probability the origin is contained in the unique infinite cluster.

We condition on the event that the origin is in the infinite cluster and consider a simple random walk on the infinite cluster, starting at the origin.

Basic question

Let $d \geq 2$ and let N be a large number. What is the probability that the walker will hit $\{N\} \times \mathbb{Z}^{d-1}$ before it hits $\{-N\} \times \mathbb{Z}^{d-1}$?

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Quenched question: Is it true that if N is large enough, then with very high probability the configuration is so that this probability is very close to $\frac{1}{2}$?

Main Theorem

The question above is answered using the following theorem:

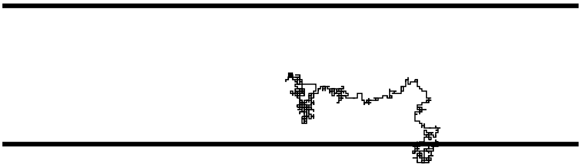
Main Theorem: Let $d \geq 2$ and let ω be a configuration s.t. the origin is in the infinite cluster. Let $(X_n)_{n\geq 0}$ be the random walk starting at the origin. Let

$$\widetilde{B}_n(t) = \frac{1}{\sqrt{n}}(X_{\lfloor tn \rfloor} + (tn - \lfloor tn \rfloor)(X_{\lfloor tn \rfloor+1} - X_{\lfloor tn \rfloor})), \qquad t \geq 0.$$

be its scaled linear interpolation. Then for all $T > 0$ and for P_0-almost every ω, $(\widetilde{B}_n(t) \colon 0 \leq t \leq T)$ converges in law to a d-dimensional isotropic Brownian motion $(B_t \colon 0 \leq t \leq T)$ with a positive diffusion constant depending only on the percolation parameter p.

Back to basic question

Since Brownian Motion hits $\{N\} \times \mathbb{Z}^{d-1}$ before it hits $\{-N\} \times \mathbb{Z}^{d-1}$ with probability $\frac{1}{2}$, we get that for most configuration $\{N\} \times \mathbb{Z}^{d-1}$ will be hit before $\{-N\} \times \mathbb{Z}^{d-1}$ with probability very close to $\frac{1}{2}$.



Remark

The same result has been independently and simultaneously proven by Mathieu and Piatnitski. Their methods are different.

Related results

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- Rassoul-Agha and Seppäläinen (2004) proved quenched convergence to Brownian Motion for a general class of random walk in space-time random environments.
- Barlow (2004) proved quenched Gaussian estimates for the heat kernel of the walk on percolation clusters.

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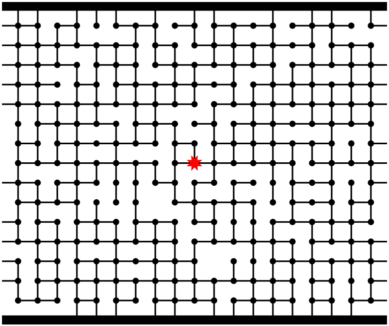

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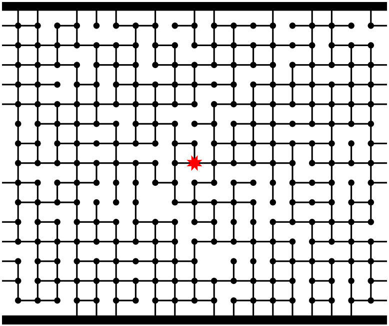

This is a harmonic function, so we solve the linear equations.

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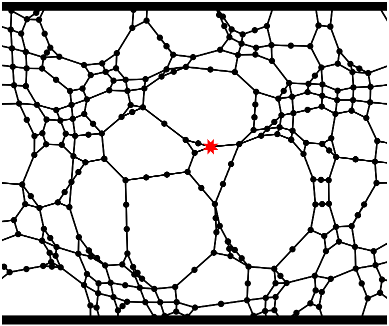

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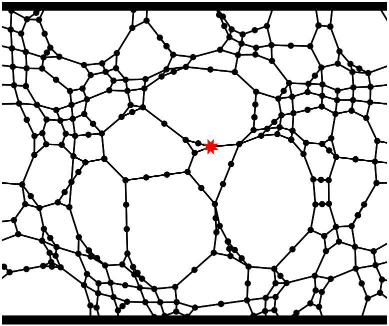

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The walk on the deformed lattice is a martingale. If it is an L^2 martingale satisfying the conditions of the Lindeberg-Feller Theorem, then it converges to Brownian motion.

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- The increments of χ are shift invariant, i.e. for every $x, y, t \in \mathbb{Z}^d$ and $\omega \in \Omega^*$, we have

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- $(\chi_\omega(x) - \chi_\omega(y)) \cdot \omega(\langle x, y \rangle) \in \mathrm{L}^2$

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Natural candidate: Let $\{X_n^{(x)}\}$ be the random walk starting at x. If

$$\lim_{n\to\infty}\left[E\left(X_n^{(x)}\right)\right]$$

exists then it satisfies harmonicity and shift-invariance, and we can take $\chi(x)$ to be its difference from x.
Problem: We don't know how to prove convergence.

The Corrector

However, following the arguments of Sidoravicius and Sznitman (2003) and Kipnis and Varadhan (1986) one can prove that

$$\phi(x) := \lim_{n\to\infty}\left[E\left(X_n^{(x)}\right) - E\left(X_n^{(0)}\right)\right]$$

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Therefore the only missing ingredient is that $\chi(x)$ is small with respect to x.

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Proposition 2: For $d \geq 2$, In $\mathbb{Z}^d$, for every ϵ,

$$\lim_{n\to\infty} \frac{1}{n^d} \# \{x : \chi(x) > \epsilon n\} = 0$$

ω-almost surely.

Why are these propositions sufficient?

Proposition 1 is sufficient because with very high probability

$$\max_{1\leq n\leq T}\|\chi(X_n)\| = o\left(\max_{1\leq n\leq T}\|X_n+\chi(X_n)\|\right) = o(\sqrt{(}T)).$$

Proposition 2 is sufficient because using Barlow's bound, with very high probability for **most times** we are in a vertex x such that

$$\|\chi(x)\| \ll \|x\|.$$

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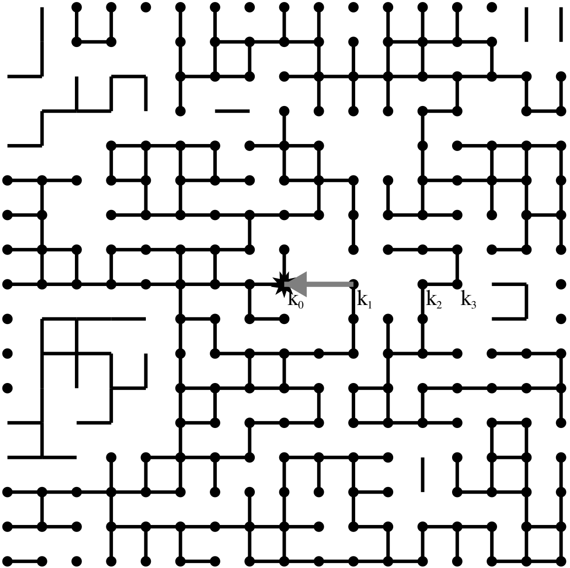
Then $\sigma_e : \Omega^\star \to \Omega^\star$ is measure preserving and ergodic.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k_1
k_2
k_3

The corrector is small along coordinate lines

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Then $E(F) = 0$,

and we get

$$\frac{1}{n}\chi(k_i) = \frac{1}{n}\sum_{j=1}^{i} F(\sigma_e^j(\omega)) \to 0 \quad \text{a.s.}$$

By the pointwise ergodic theorem.

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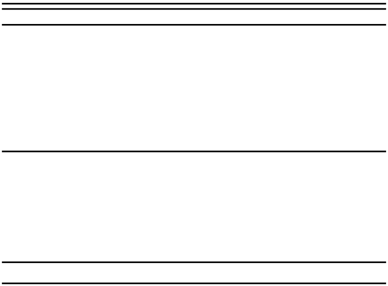
If K is large enough, then a line is nice with positive probability.

Proof of two-dimensional case

By the ergodic theorem, there are many nice lines. In particular, the **spacing** between nice lines is sublinear.



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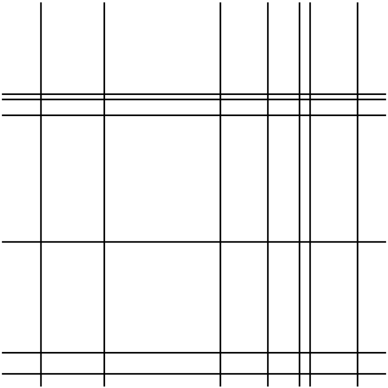
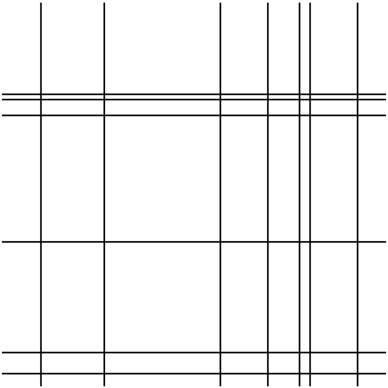


The corrector along nice lines

Along the nice lines, the value of the corrector is bounded by $2K + 2\epsilon n$.



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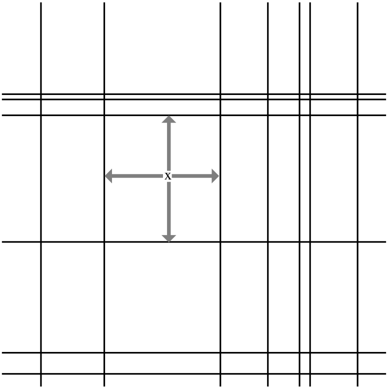
The function $x + \chi(x)$ is harmonic. Therefore, by the maximum principle, χ is bounded by $2K + 2\epsilon n +$ maximum spacing .



The corrector between nice lines

Since the spacing is sublinear, for n large enough we get that

$$\max_{x\in[-n,n]^2} |\chi(x)| < 2K + 3\epsilon n,$$

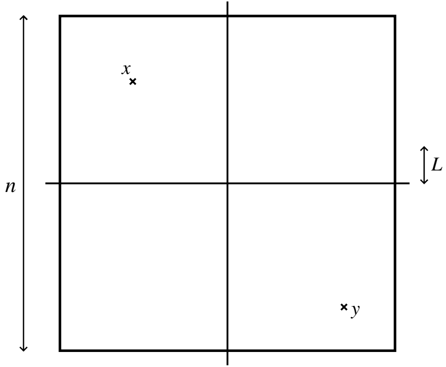
as desired.

$\square$

Proof of higher dimensional case

We want to show that for most pairs x and y,

$$|\chi(x) - \chi(y)| < \epsilon n$$



Proof of higher dimensional case

We do so using induction:

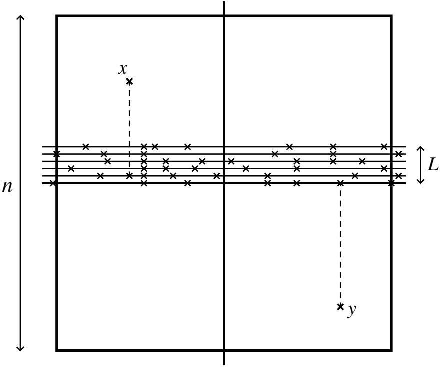
Let d be the dimension of the space, and let $e_1, \ldots, e_d$ be the standard basis.

We use induction on k to show that the proposition holds for $\text{span}(e_1, \ldots, e_k)$ for $k = 1, 2, \ldots, d$.

The base case $k = 1$ follows from the ergodic theorem.

Proof of higher dimensional case
Induction Step

We stack a fixed number L of hyperplanes of dimension $k-1$. The statement holds for all of them. The vast majority of lines parallel to e_k are nice, and intersect $\mathcal{C}_\infty$ on one of the L hyperplanes.



Open problems

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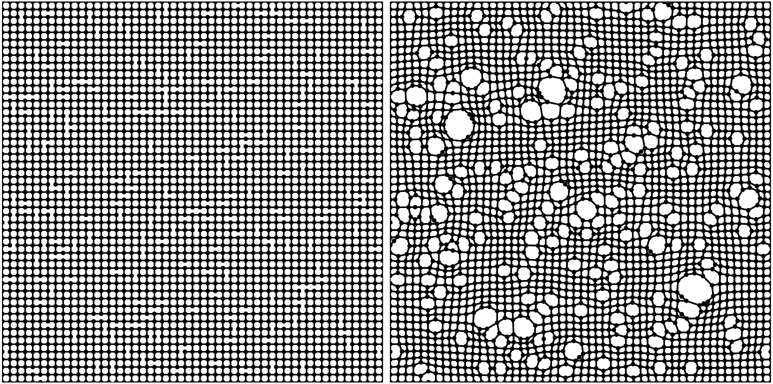
2. Are there harmonic functions of sub-linear growth on a percolation cluster?

Playing with the corrector

Percolation cluster and its deformation: $p = 0.95$



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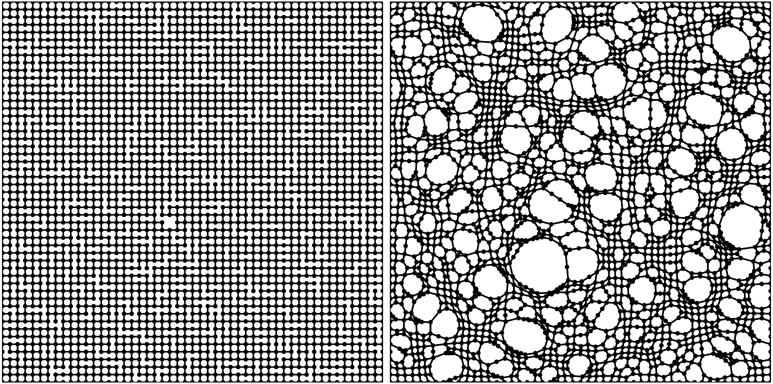
Percolation cluster and its deformation: $p = 0.85$



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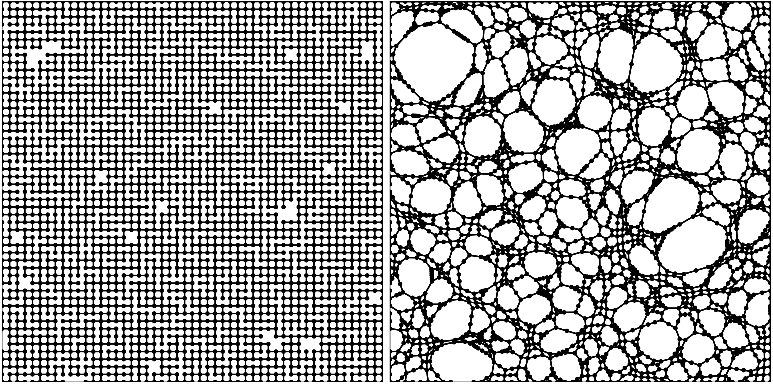
Percolation cluster and its deformation: $p = 0.75$



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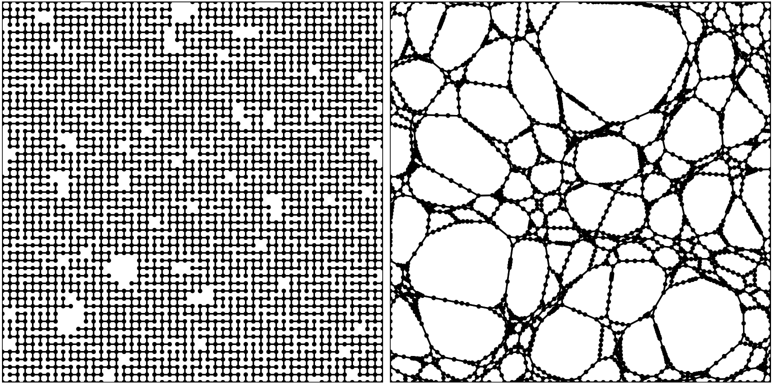

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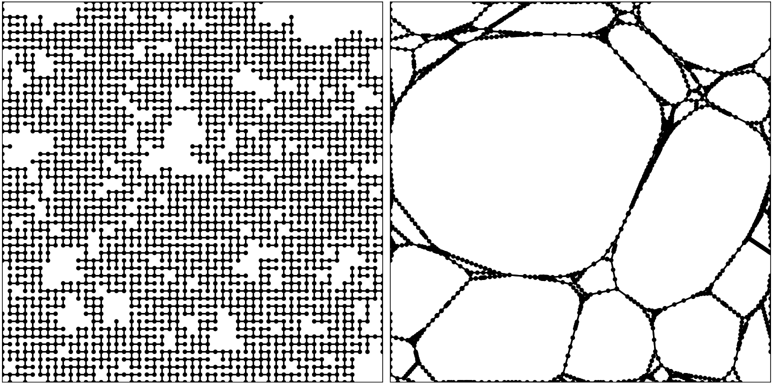

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