Biased random walk on percolation clusters

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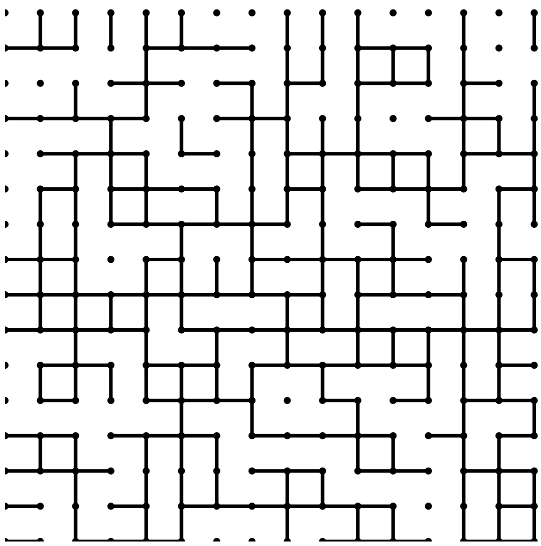
Related paper:

[Berger, Gantert & Peres]
(Prob. Theory related fields, Vol 126,2, 221–242)

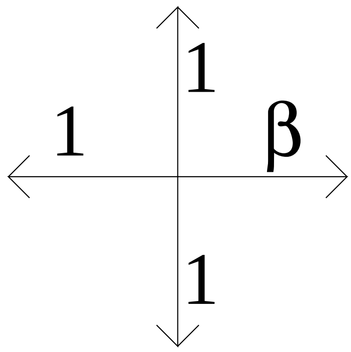
The Model

Percolation on $\mathbf{Z}^2$: Choose $0 < p < 1$, and declare each edge of $\mathbf{Z}^2$ **open** with probability p and **closed** with probability $1 - p$. We do it so that different edges are independent of each other.

If $p > \frac{1}{2}$ then, a.s., there exists a unique infinite open cluster. We only focus on this phase.



Random walk biased to the right: Choose some $\beta > 1$. With probability proportional to β, the walker tries to walk to the right, and with probability proportional to 1, it tries to go to any other direction.



We condition on the event that the origin is in the infinite cluster, and start the walk at the origin

Previous results on unbiased walk on percolation clusters (partial list):

Transience vs. Recurrence:

Grimmet, Kesten, Zhang (1993)

Benjamini, Pemantle, Peres (1998)

Haggstrom, Mossel (1998)

Angel, Benjamini, Berger, Peres (2002)

Return Probabilities:

Heicklen, Hoffman (1999)

Mathieu, Remy (2003)

Barlow (2003)

Mixing times:

Benjamini, Mossel (2002)

Scaling limit:

Sidoravicius, Sznitman (2003)

Results:

Theorem: Fix $\frac{1}{2} < p < 1$. Let $Z_n = (X_n, Y_n)$ be the location at time n. Then,

1. For every $\beta > 1$, a.s. the walk is transient to the right, i.e.

   $$\lim_{n\to\infty} X_n = \infty$$

   and the speed

   $$S(\beta) = (S_x(\beta), S_y(\beta)) = \lim_{n\to\infty} \frac{Z_n}{n}$$

   exists and is a constant a.s.

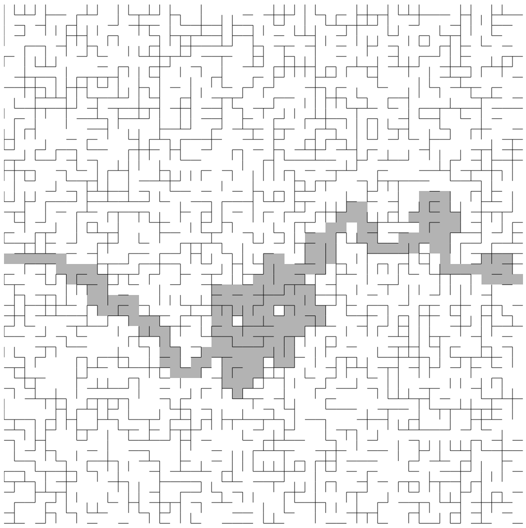
2. If β is large then $S(\beta) = 0$.

3. If β is small then $S_x(\beta) > 0$.

Conjecture: There exists $1 < \beta_c < \infty$ s.t. if $\beta > \beta_c$ then 2 holds and if $\beta < \beta_c$ then 3 holds.

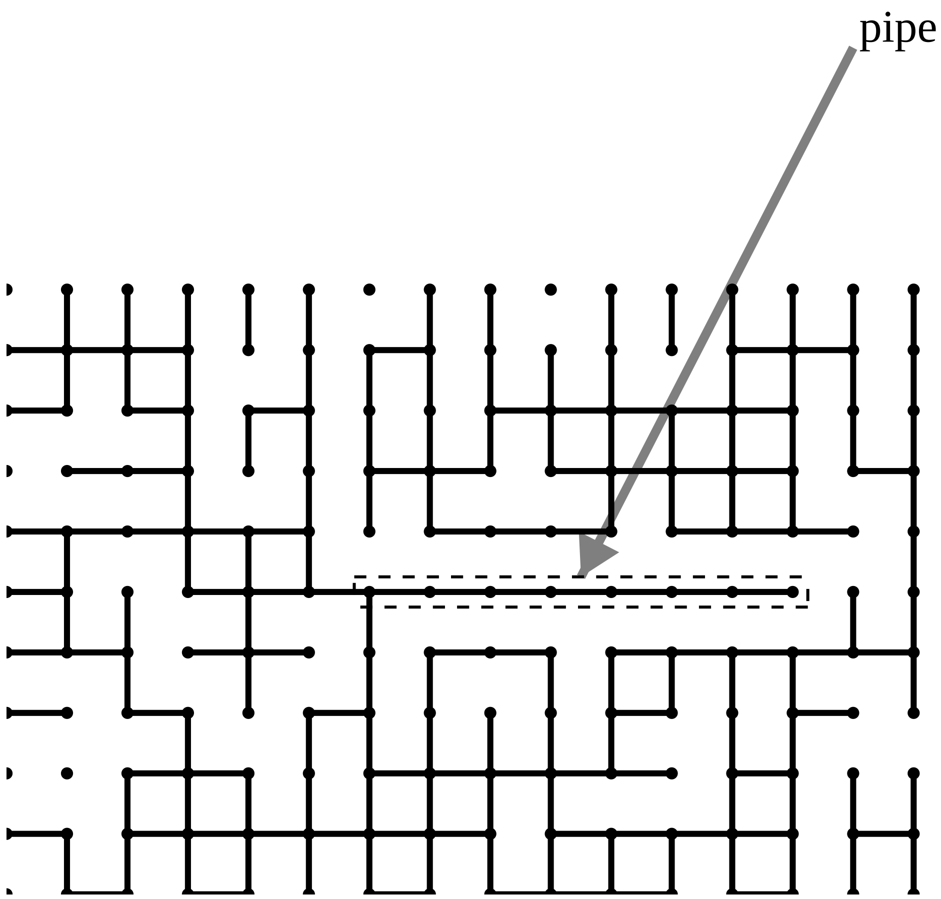
Remark: In an independent and simultaneous work Sznitman (to appear, CMP) obtained similar results for **all** dimensions.

The path of the walker:



On a different scale

Proof of zero speed for large β



If the walker entered a pipe of length n, then the expected exit time from the pipe is β^n. The frequency of the appearance of pipes of length n is $(1-p)^{2n+1}p^n$. Therefore, for every n the speed is bounded from above by

$$\frac{(1-p)^{-1}}{\left(\beta p(1-p)^2\right)^n} \tag{1}$$

and (1) approaches zero for $\beta > p^{-1}(1-p)^{-2}$.

Plan of the proof of positive speed for small β:

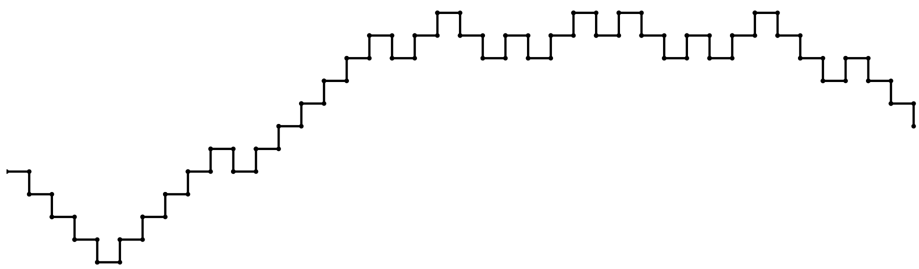
1. For p close enough to 1:
   (a) Definition of good points, bad points and traps.
   (b) Time spent in traps.
   (c) Transience to the right.
   (d) A priori bound for progress.
   (e) Definition of regenerations and time between regenerations.

2. For all $\frac{1}{2} < p < 1$: Renormalization.

Good point: We say that $z = (x, y)$ is a **good point** if there exists an open path

$$z = z_0, z_1, z_2, z_3, \ldots$$

s.t. $z_{2k+1} - z_{2k} = (1, 0)$ and $z_{2k} - z_{2k-1} = (0, \pm 1)$.



Bad points: We say that z is a **bad point** if it is in the infinite cluster but it is not a good point.

Traps: A **trap** is a connected component of bad points.

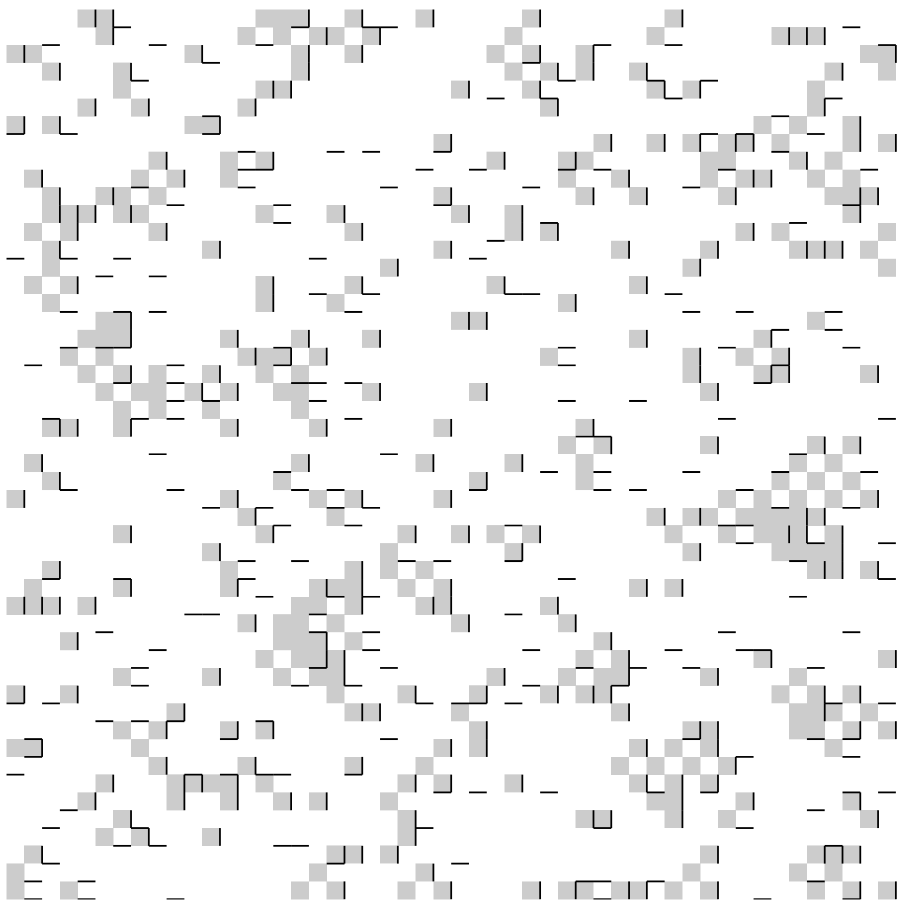
Lemma: Let T be the trap containing 0. If p is close enough to 1, then the **length**

$$\max\{x_1 - x_2 \ : \ (x_1, y_1), (x_2, y_2) \in T\}$$

and the **width**

$$\max\{y_1 - y_2 \ : \ (x_1, y_1), (x_2, y_2) \in T\}$$

of T both have exponential tails.



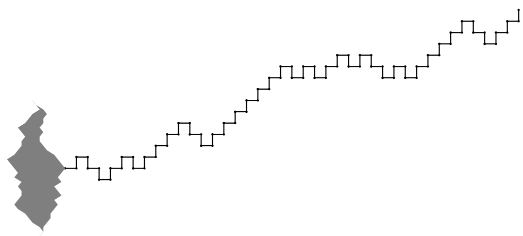
In particular, a point has a positive probability of being good.

Time spent in traps:

Lemma: Let T be a trap whose length and width are bounded by k. Then the expected time spent in T is no more than β^{2k}.

Number of visits to a point:

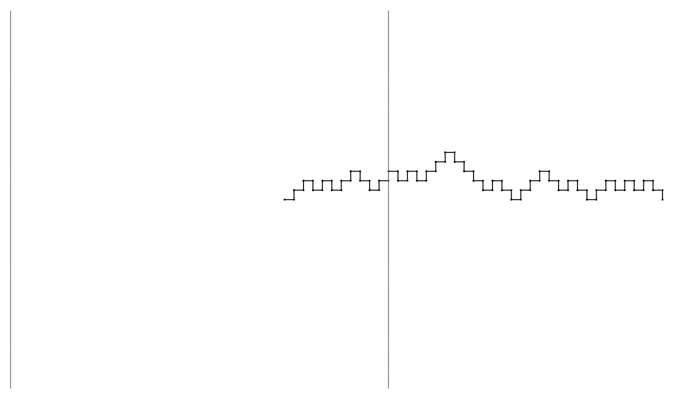
Conditioned on having visited a point z, the number of visits to z is distributed geometrically.



Lemma: Let T be the trap including z, and assume that its length and width are bounded by k. Then, the expected number of visits to z is bounded by $C\beta^{2k}$.

Transience to the right:

Lemma: Assume $z = (x, y)$ is a good point, and let l be a positive number. Starting from z, with probability $1 - e^{-\Omega(l)}$ the walker will hit $\{x + l\} \times \mathbf{Z}$ before it hits $\{x - 3l\} \times \mathbf{Z}$.



Iterating this lemma, with the value of l growing, we get transience to the right.

An a priori bound for the progress to the right:

Lemma: If β is close enough to 1, then for n large enough, with probability bigger than $1 - n^{-2}$, we have

$$X_n > n^{\frac{1}{10}}.$$

Why?

Consider the square $R_n = [-n, n]^2$. Up to time n the walker does not leave R_n.

With Probability $1 - n^{-3}$, The biggest trap in R_n is no bigger than $u \log n$.

Therefore, no point is visited more than $\beta^{6u \log n}$ times. But,

$$\beta^{6u \log n} < n^{\frac{1}{20}}$$

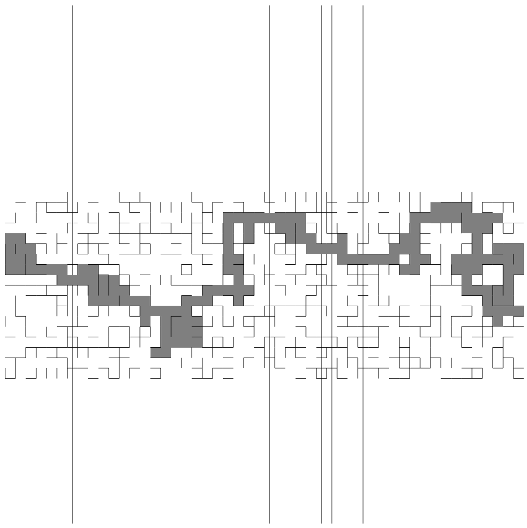
for β small enough.

Therefore, we visit at least $n^{\frac{19}{20}}$ points.

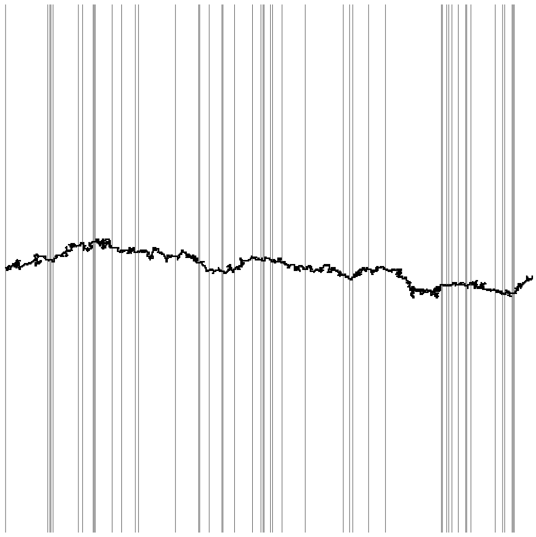
Now, the lemma follows from Varopoulos–Carne's bound.

Regenerations

We say that a **regeneration** occurred at time n if $X_k < X_n$ for all $k < n$ and $X_j > X_n$ for all $j > n$.



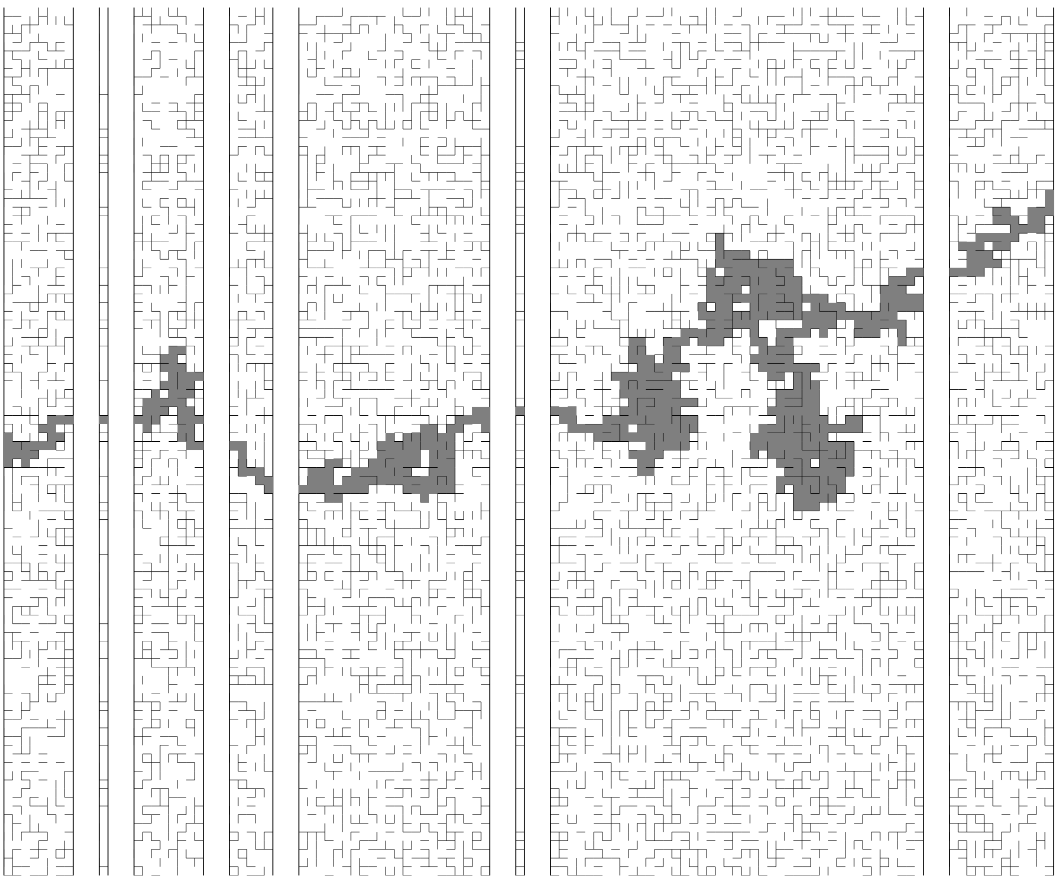
On a different scale



Lemma: Almost surely, infinitely many regenerations occur.

Proof: Because of transience to the right and the Markov property of the walk given the configuration.

Lemma: Let T_i be the time of the i-th regeneration. The segments of the walk $\{Z_t : T_i \leq t < T_{i+1}\}$ form an i.i.d. process.



Finishing the proof

All we have to show is that

$$E(T_{i+1} - T_i) < \infty. \tag{2}$$

We show $E(T_1) < \infty$.

(2) follows from a similar argument.

We want to show that

$$\sum_{n=1}^{\infty} \mathbf{P}(T_1 > n) < \infty.$$

We estimate $\mathbf{P}(T_1 > n)$ for big n.

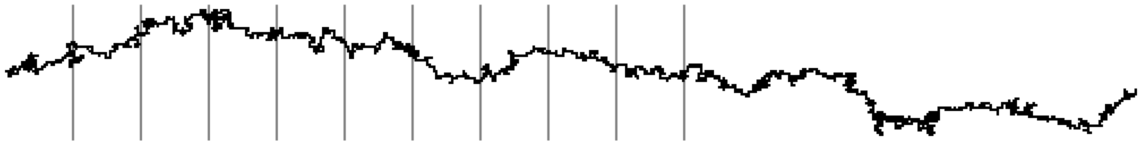
Estimating $\mathbf{P}(T_1 > n)$:

$\mathbf{P}(T_1 > n)$ is the probability that no regeneration happened until time n.

Let τ_k be the first time t such that

$$X_t >= k(\log n)^2.$$

With probability bigger than $1 - n^{-2}$, for every $k < n^{1/20}$, $\tau_k < n$.



Lemma: There exists a constant ρ s.t. for every n large enough and every $k < n^{1/20}$, there exists an event E of probability bigger than $1 - n^{-2}$ s.t. conditioned on E and on the event that there was no regeneration at $\tau_j, j < k$, the probability of regeneration at τ_k is at least ρ.

By this lemma,

$$\mathbf{P}(T_1 > n) < 2n^{-2} + (1 - \rho)^{n^{1/20}}$$

□