Yosi and Me

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- 6 on topological aspects of solid state physics;
- 3 on miscellaneous subjects.

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And his sunny disposition and pleasant personality added to the joy.

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Prior to our work, virtually all of the mathematically centered work focussed on the self-adjointness question.

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This led me to conjecture and then prove, with an assist from Ed Nelson, that

$$\left| e^{-tH(\vec{a},V)} \varphi \right| \leq e^{-tH(\vec{a}=0,V)} \left| \varphi \right|$$

something I named diamagnetic inequalities.

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He first convinced Ira to join him and then involved me.

Aside on Electric Fields

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One lovely result is the Avron–Herbst formula,

$$H_o = p^2 + x_1 \Rightarrow e^{-itH_0} = e^{-itx_1} e^{-itp^2} e^{it^2 p_1} e^{-it^3/3}$$

a formula with interesting physics built into it.

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- Ground State has $m = 0$ (via FKG Inequalities; needs monotonicity of V) for Hydrogen Zeeman and some other models;
- Reduction of Center of Mass (subtle because non-commuting total "momenta.")

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- Magnetic Bottles, even in odd dimension
- Conjecture (proved by Lieb) that for constant field for bosons, ground state of $H(a, V) + \vec{\sigma} \cdot \vec{B}_0$ goes down for $\vec{B}_0 \neq 0$ compared to $B_0 = 0$.

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This proof was not rigorous in Yosi's 1981 paper but was made rigorous by Helffer-Sjöstrand in 1985.

Paramagnetic Conjectures

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The paramagnetic conjecture of Hogreve, Schrader, and Seiler says that this holds for non-constant fields.

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The idea is the same as that behind the Bohm–Aharanov effect. If, due to an infinite V in the region where $B \neq 0$, the electron doesn't feel $\vec{B}$ directly, it can still feel $\vec{a}$. But then $\langle \varphi, \vec{B} \cdot \vec{\sigma} \varphi \rangle = 0$ so one can't have opposite inequalities.

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There is a story behind that.

Almost Periodic Jacob and Schrödinger Operators

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We decided there wasn't that much in the second subject so we'd finish it off in six months and then turn to the quasiclassical limit.

Almost Periodic Stuff

Were we wrong! Thirty years later, I'm still thinking about the subject of almost periodic operators and never did get to the quasiclassical limit. So instead we wrote the founding documents of another industry, so much so that the Newton Institute will have a special term on it in the first half of 2015.

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The subject was certainly "in the air." Shortly before us Moser and Johnson, first separately, but then jointly, looked at some of the subjects we did.

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Johnson–Moser used ODE methods. Yosi and I focused on functional analytic methods. Here are some of our results:

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- Theory of Rings of Saturn which nature rudely rejected.

Almost Matthieu Operator

- We named

  $$(hu)(n) = u(n+1) + u(n-1) + \lambda \cos(\pi \alpha n + \theta) u(n)$$

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- First a.p. models with singular continuous spectrum (Diophantine vs. Liouville frequencies).

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I would be remiss in leaving the subject of the almost Matthieu equation without mentioning the progress of Avila, Jitomirskaya, and Last (ten Martini problem).

Topological Invariants

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Geometry in physics has been a constant theme in Yosi's work, not only in the joint work we did from 1983 to 1994 but also since then.

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- (with Ruedi Seiler again) The index of a pair of projections reconciling the Chern class (TKNN) and the noncommutative differential geometry (Connes–Bellissard) views of the quantum Hall effect.

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- (with Ruedi Seiler again) The index of a pair of projections reconciling the Chern class (TKNN) and the noncommutative differential geometry (Connes–Bellissard) views of the quantum Hall effect.

For the first we promised, but never produced, a longer paper.

TKNN Integers

In trying to understand the integer quantum Hall effect, Thouless, Kohmoto, Nightingale, and den Nijs were able to compute the conductance due to a single non-degenerate band to a constant times an integer, which we called TKNN integers (Avron–Seiler–Simon–1983).

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They had essentially rediscovered the Chern class.

Our paper includes an explicit formula that Yosi loved writing down:

$$n_j = \frac{i}{2\pi} \int \mathrm{Tr}((dP_j)P_j(dP_j))$$

where $P_j(k)$ is the projection onto the jth base function at k and n_j is the TKNN integer.

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We did this with the exact sequence of a fibration. There is an interesting story behind our use of this tool.

Berry's Phase for Fermions

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This is true for bosons but not for fermions. For bosons, time reversal, T, is an antilinear map with $T^2 = 1$, so a complex conjugation.

But for fermions, $T^2 = -1$, so such systems have a quaternionic structure and there will be, in general, a nontrivial Berry and TKNN structure.

Berry's Phase for Fermions

We found a number of abstract results, for example, if $M_n(\mathbb{C})$ is the $n \times n$ complex Hermitian matrices with simple eigenvalues, the only non-trivial homotopy group is $\pi_2(M_n(\mathbb{C})) = \mathbb{Z}^{n-1}$, corresponding to $n-1$ TKNN integers. For quaternion $\mathbb{H}$, the only low-dimensional nontrivial π is $\pi_4(M_n(\mathbb{H})) = \mathbb{Z}^{n-1}$.

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We also did a rather complete analysis of spin 3/2 in a quadruple field, a beautiful and subtle model with SO(5) symmetry.

Yosi, the Zen Master

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Abstract: Yes, but some parts are reasonably concrete.

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Abstract: Yes, but some parts are reasonably concrete.

There is even a story on how we got it included.

The Relative Index for Projections

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We used this to contrast the TKNN theory of the quantum Hall effect and an approach of Bellissard.

If there is time, I'll say more about the pairs of projections soon.

Miscellaneous

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- Our first joint paper (1977) studied analytic continuation of the band functions of periodic potentials in higher dimension. The one-dimensional case had been studied by Walter Kohn using ODE methods (and we also extended his work).
- A "homework problem" of Evans Harrell. He had studied asymptotics of the gaps in the spectrum of

$$-\frac{d^2}{dx^2} + 2\kappa\cos(2x)$$

  where his formula had a constant involving integrals of Airy functions, and he asked what the constant value was. We found a new approach to the problem—a cute argument that led to a direct calculation of the constant.

Miscellaneous

- A joint paper with Jim Howland involving a long-time love of Yosi's—the adiabatic theorem—where he is a world expert. We studied adiabatic perturbations for certain cases of dense point spectrum.

Let me give some details of the proof of one of our results (in the paper with Ruedi Seiler in J. Func. Anal.):

A Tale of Two Projections

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If P and Q are two orthogonal projections so that $P - Q$ is trace class, then $\mathrm{Tr}(P - Q)$ is an integer.

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If P and Q are two orthogonal projections so that $P - Q$ is trace class, then $\mathrm{Tr}(P - Q)$ is an integer.

Ed Effros seems to be the first to prove this. He used other methods.

Supersymmetry

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We added the supersymmetry

$$AB + BA = 0$$

Supersymmetry

Following Kato, we define

$$A = P - Q, \qquad B = 1 - P - Q$$

Kato noted that

$$A^2 + B^2 = 1$$

We added the supersymmetry

$$AB + BA = 0$$

After our preprint appeared, Kato sent us an unpublished technical report with this formula, but he didn't use it for anything.

Supersymmetry

Let $\mathcal{H}_\lambda = \{\varphi | \mathcal{A}\varphi = \lambda\varphi\}, \quad d_\lambda = \dim(\mathcal{H}_\lambda)$
so by Lidskii's Theorem, if A is trace class

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$$\mathrm{Tr}(P - Q) = d_1 - d_{-1} \in \mathbb{Z}$$

Landau–Pollak Uncertainty Principle

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To illustrate the depth of a simple pair of projections, I want to discuss a lovely result of Landau–Pollak, well known in the "signal community," but which I only recently learned about. Even though it dates to 1961, when Yosi and I were kids, I'm hoping it is also new to him and that I can present this as a birthday bouquet.

Here is their result:

Landau–Pollak Uncertainty Principle

THEOREM: Let P and Q be two orthogonal projections on a Hilbert space and $\varphi \in \mathcal{H}$ with $||\varphi|| = 1$. Then

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Moreover, if Ran P + Ran Q is not dense in $\mathcal{H}$, for any α, β with $\alpha + \beta > \arccos(||PQ||)$, there is φ with $\arccos(||P\varphi||) = \alpha$, $\arccos(||Q\varphi||) = \beta$. If PQ is compact, one can also take α, β with $\alpha + \beta = \arccos(||PQ||)$.

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Below, I'll prove the inequality leaving the best possible "moreover" result to the listener.

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This gives useful information even if $\langle x^2\rangle$ or $\langle p^2\rangle$ is infinite.

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The proof of Landau–Pollak is algebraic and less than transparent. I found a lovely (he said modestly) geometric proof which it turns out is already hinted at in a 1997 review article (on the uncertainty principle) of Folland–Sitaram.

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$$||P\varphi|| = \langle\varphi, P\varphi\rangle/[||\varphi||\,||P\varphi||]$$

is the cos of the angle between φ and $P\varphi$.

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Moreover,

$$\frac{|\langle P\varphi, Q\varphi\rangle|}{||P\varphi||\,||Q\varphi||} \leq ||PQ||$$

so

$$\operatorname{ang}(P\varphi, Q\varphi) \geq \arccos(||PQ||)$$

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This proves the Landau–Pollak inequality if $\varphi, P\varphi, Q\varphi$ lie in $\mathbb{R}^2$ as a real Hilbert space (they all lie on one side of $\{\varphi\}^\perp$).

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In a complex Hilbert space, one does everything for $\langle \cdot, \cdot \rangle_r = \mathrm{Re}\langle \cdot, \cdot \rangle$ and gets the general result.

Landau-Widom

One final result I'd like to mention. As a follow-up to the earlier work by Landau, Pollak, and Slepian, Landau and Widom proved a wonderful result for the case of $P = \chi_{(-a,a)}$ in x-space and $Q = \chi_{(-b,b)}$ in k-space.

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The eigenvalues of $K_c = P_a Q_b P_a$ only depend on $c = 4ab$ and the phase space volume is c. IF $N(\alpha < K_c < \beta)$ is the number of eigenvalues of K_c between α and β, they prove (quasiclassical limit as $c \to \infty$) that for all $\alpha > 0$,

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$$N(K_c > 1 - \alpha) = \frac{c}{2\pi} + O(\log c)$$

$$N(1 - \alpha > K_c > \alpha) = O(\log c)$$