Quantum statistics on graphs

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J^3 2011
J^3 + Adam Sawicki 2013 preprint

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Quantum statistics

The standard formulation is to take

$$\mathcal{H}_n = \mathcal{H} \otimes \cdots \otimes \mathcal{H},\ n\text{-particle Hilbert space},$$

and restrict to

$$\sigma|\Psi\rangle = (\,\mathsf{sgn}\,\sigma)^{\epsilon}|\Psi\rangle, \sigma \in S_n$$

$$\epsilon = \begin{cases} 0, & \text{Bose statistics} \\ 1 & \text{Fermi statistics} \end{cases}$$

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Are there other possibilities?

Quantum mechanics on non-simply connected spaces

M, manifold, with fundamental group $\pi_1(M)$.

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$$H_1(M) \simeq \mathbb{Z}^p \oplus (\mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_q)\,,$$

where d_j divides d_{j+1}. $\mathbb{Z}^m$ is free component, and $\mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_q$ is torsion component.

Repn's of $H_1(M)$ are classified by p free phase factors $e^{2\pi i \alpha_j}$ and q discrete phase factors $e^{2\pi i a_k/d_k}$.

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$M = \mathbb{R}^2 - \{\mathbf{0}\}$, punctured plane
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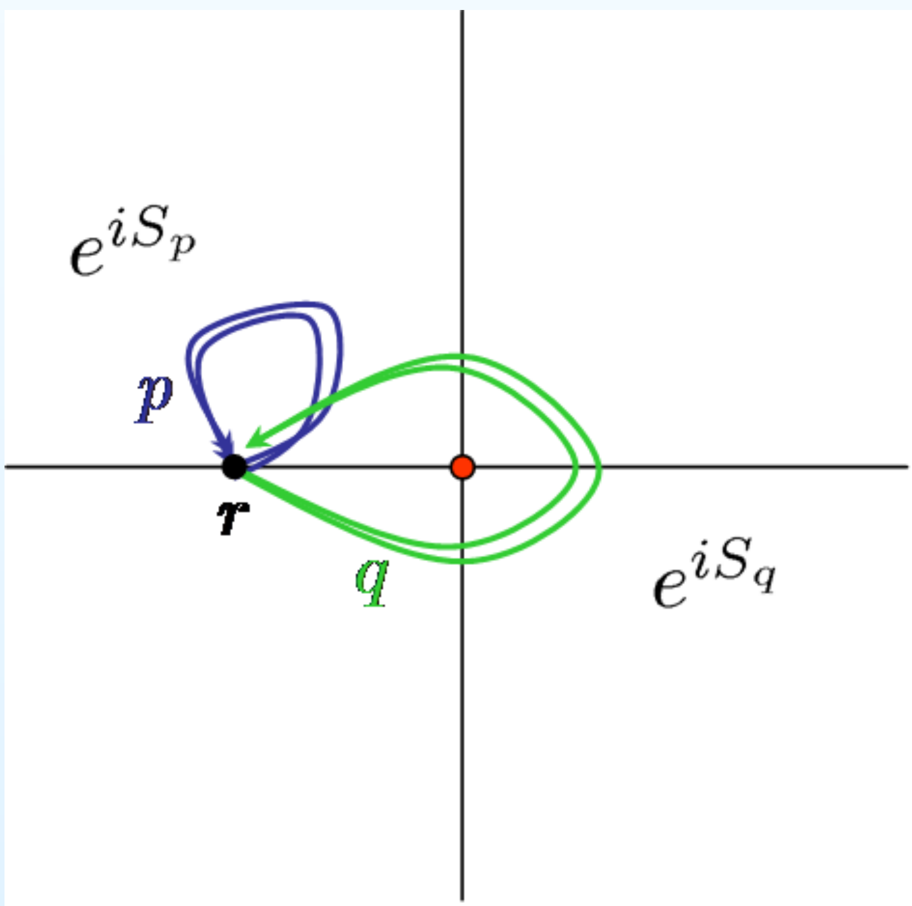

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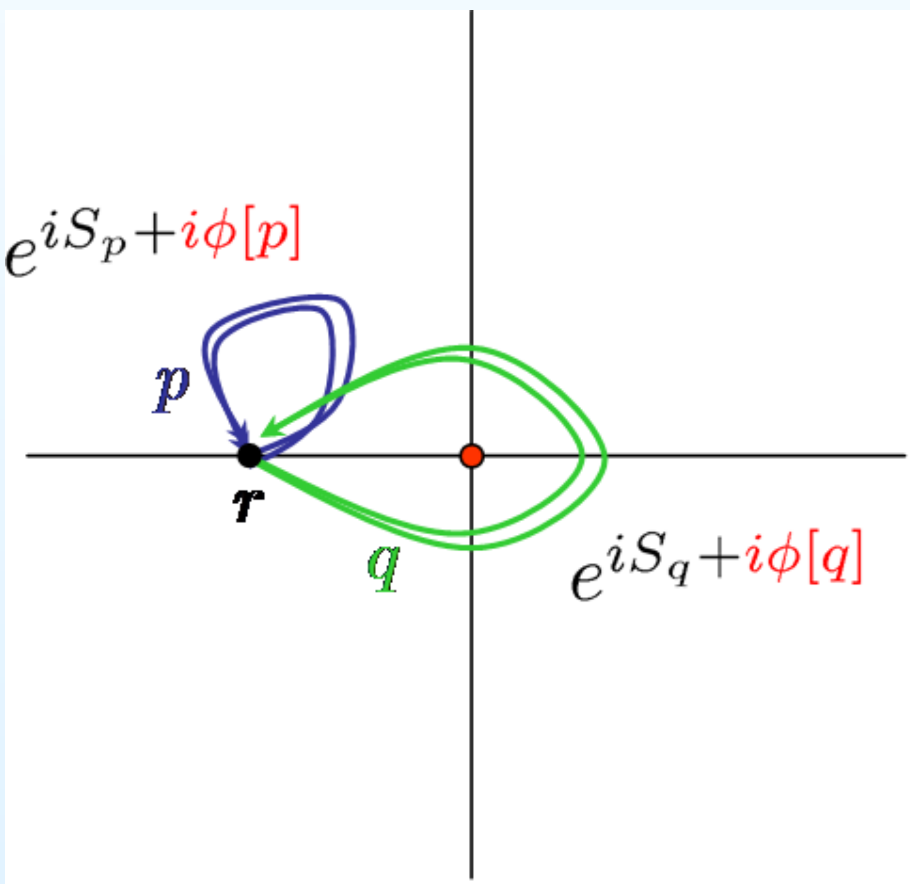

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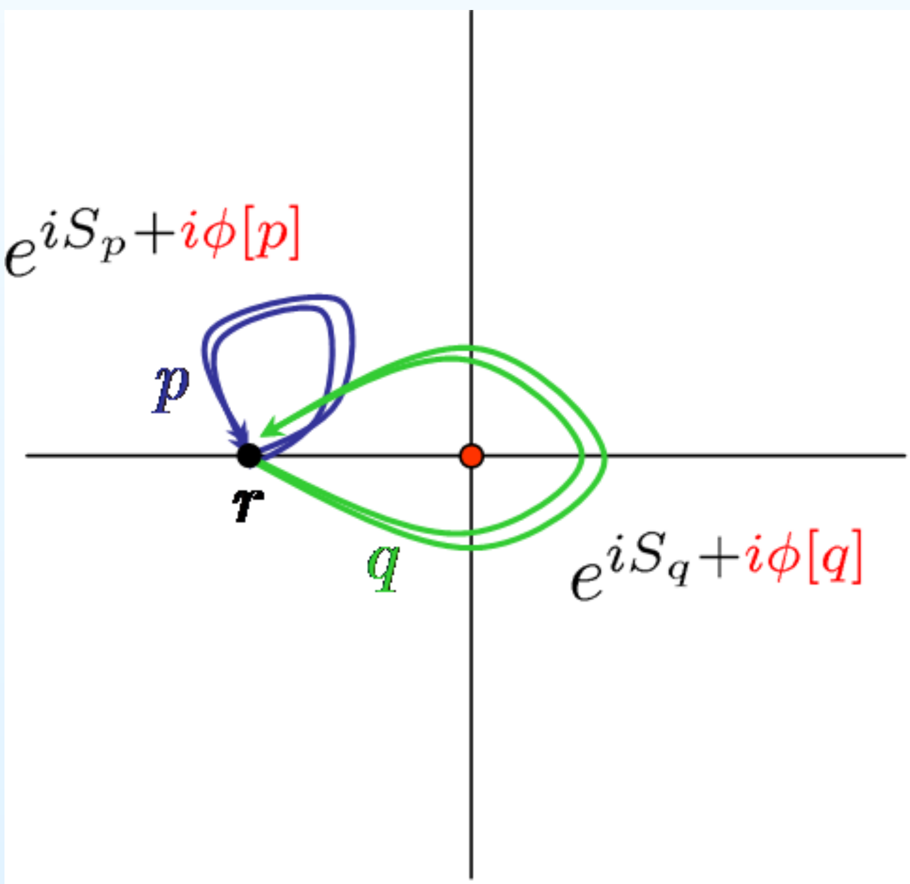

Unitarity and composition properties of the propagator imply that

$$\phi[p] = \alpha \times \text{winding number}(p)$$

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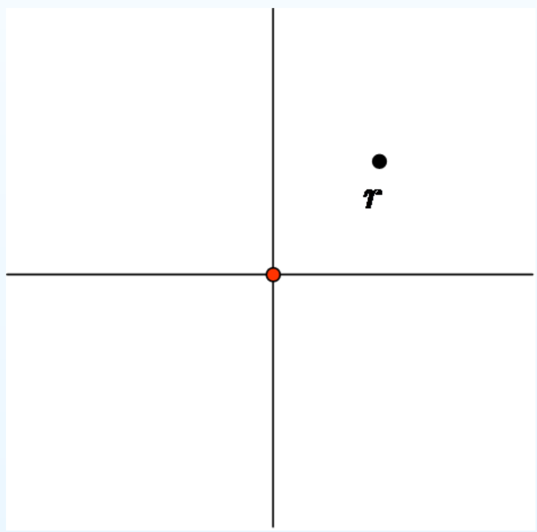

$$\mathcal{H} = L^2(\mathbb{R}^2 - \{\mathbf{0}\}),\ \text{Hilbert space}$$

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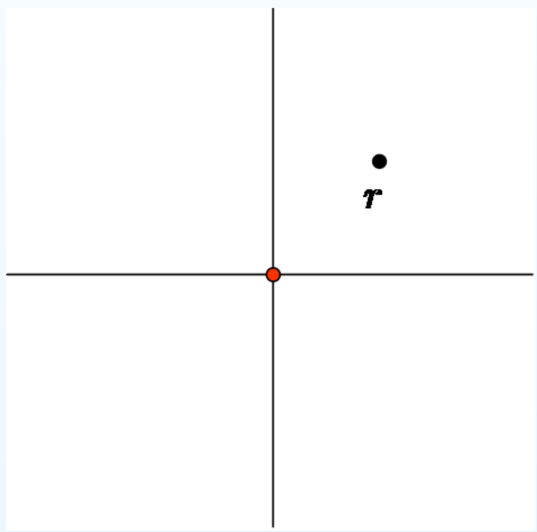

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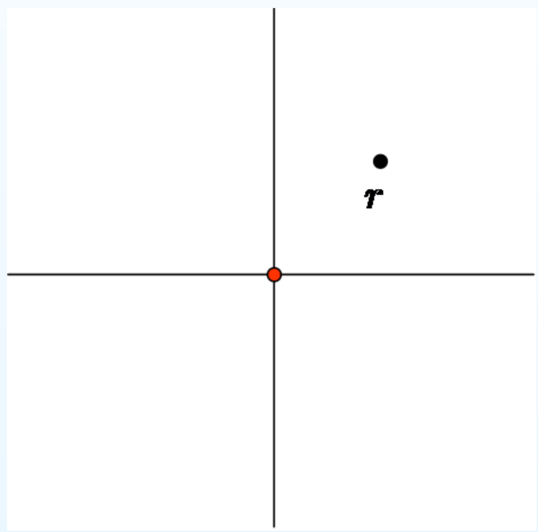

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$$\mathbf{A} = \alpha \boldsymbol{\nabla}\theta \Rightarrow \psi(\mathbf{r}) \sim r^{\min(\tilde{\alpha}, 1-\tilde{\alpha})}, \text{ change of domain}$$

Quantum statistics

Laidlaw and DeWitt (1971), Leinaas and Myrheim (1977)

X, one-particle configuration space

$$C_n(X) = (X^n - \Delta_n)/S_n,$$

configuration space of n indistinguishable particles, with coincident configurations Δ_n removed

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- $X = \mathbb{R}^2$: $\pi_1(C_n) =$ braid group, $H_1(C_n) = \mathbb{Z}$
  ($C_2 \sim \mathbb{R}^2 - \{\mathbf{0}\}$)
  Anyon statistics, $e^{i\alpha}$

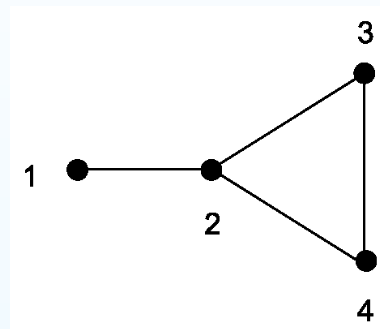
What about many-particle graphs?

Combinatorial and metric graphs

G, combinatorial graph

$V = \{1, \dots, N\}$, vertices

$E = \{\{j, k\}\}$, edges (undirected)



Take G to be simple (no loops or parallel edges) and connected.

A, adjacency matrix

$$A_{jk} = \begin{cases} 1, & e(j,k) \in E, \\ 0, & \text{otherwise}, \end{cases}$$

symmetric off-diagonal, $(A^p)_{jk} \neq 0$ for p large

d_j, degree of vertex j

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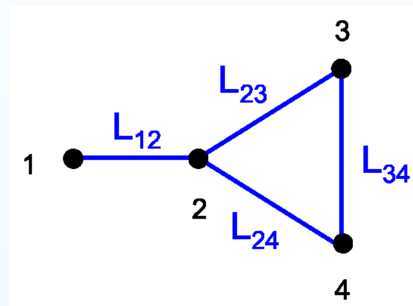

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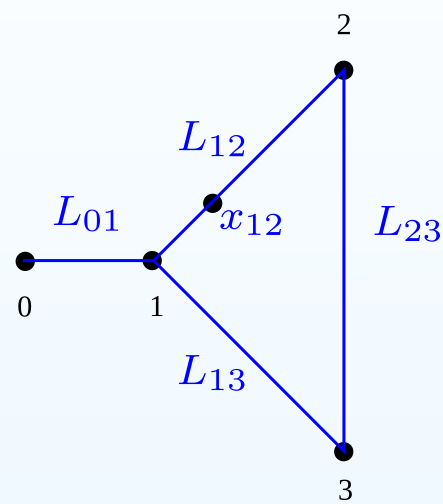
d_j, degree of vertex j
Associate an interval $[0, L_{jk}]$ to each edge $\{j, k\}$.

Identify endpoints with coincident vertices.

$$\Gamma = \sqcup_i I_i / \sim,$$

1-dimensional cell complex

Quantum metric graph



$\Psi = \{\psi_e(x_e)\}$, wavefunction

$H = \sum_e \left(-i\frac{d}{dx_e} - A_e(x_e)\right)^2 + \phi_e(x_e)$, Hamiltonian

Boundary conditions on ψ_e's required to make H self-adjoint

Eg, Neumann conditions,

$$\psi_e \text{ continuous at vertices,}$$

$$\sum_{e|j\in e} \psi_e'(j) = 0, \text{ sum of outgoing derivatives vanishes}$$

See, eg, Berkolaiko and Kuchment 2013

n-particle quantum metric graphs

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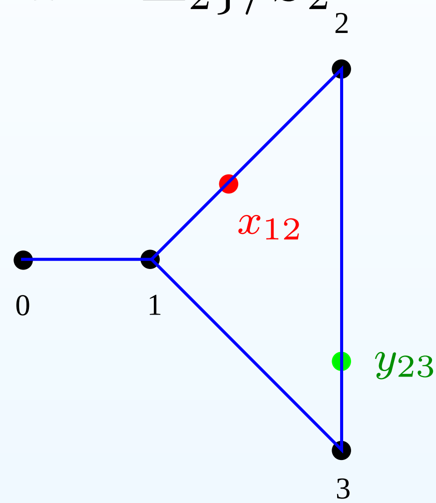
$\Gamma_2 := C_2(\Gamma) = \{\Gamma \times \Gamma - \Delta_2\}/S_2$



$\Psi = \{\psi_{ef}(x_e, y_f)\}$, wavefunctions

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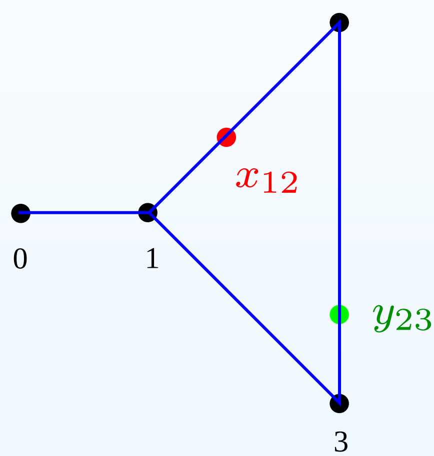

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$$H_{ef} = -\left(\frac{\partial^2}{\partial x_e^2} + \frac{\partial^2}{\partial y_f^2}\right)$$

Require boundary conditions which render H self-adjoint.
Then try to incorporate rep'n of $\pi_1 \ldots$

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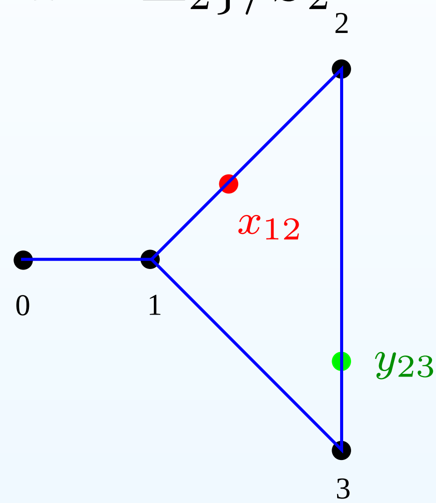

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The topology is more easily incorporated in the combinatorial setting. . .

Quantum combinatorial graphs

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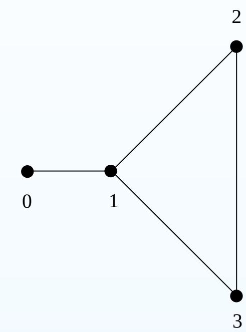

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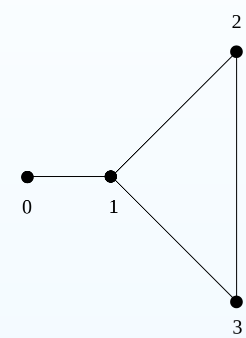

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$|\Psi\rangle = \sum_j \psi_j |j\rangle \in \mathbb{C}^N$, N-dimensional Hilbert space
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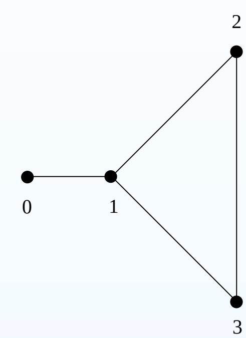

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H, $N \times N$ hermitian matrix, Hamiltonian

Eg, $KE = A - D$, $D_{jk} = v_j \delta_{jk}$, discrete Laplacian

$H_{jk} = 0$ unless $j = k$ or $A_{jk} = 1$

Short-time dynamics involves transitions to adjacent vertices.

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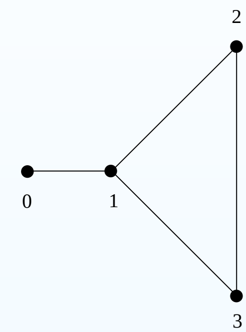

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Hamiltonians can be parameterised by 1-dimensional repn's of the first homology group. . .

π_1 and H_1 for combinatorial graph

For a combinatorial graph $G \ldots$

$(j_0, \ldots, j_p)$, path, sequence of adjacent vertices

$c = (j_0, \ldots, j_p = j_0)$, cycle on G, starts and ends at j_0

$\mathcal{C}(G, *)$, cycles which start and end at $*$

Regard cycles which differ by retracings as equivalent.

$$\pi_1^c(G) = \mathcal{C}(G, *)/ \sim,$$

fundamental group of G.

Unchanged by adding/removing vertices of degree 2.

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$$\pi_1^c(G) \cong \pi_1(\Gamma)$$

$\pi_1^c(G)$ is the free group on β elements, where

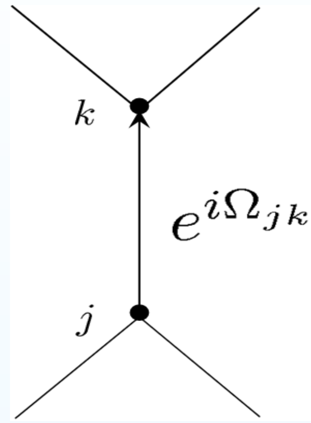
$$\beta = |E| - |V| + 1$$

$$H_1^c(G) = \mathbb{Z}^\beta$$

Gauge potentials

$$H \to H(\Omega)$$

$$H_{jk} \to e^{i\Omega_{jk}} H_{jk}$$



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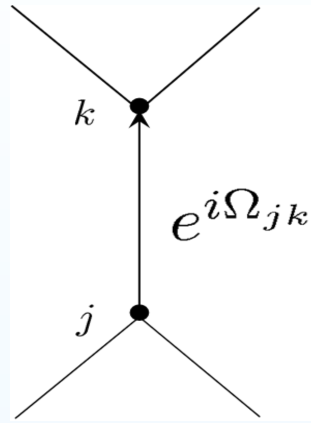

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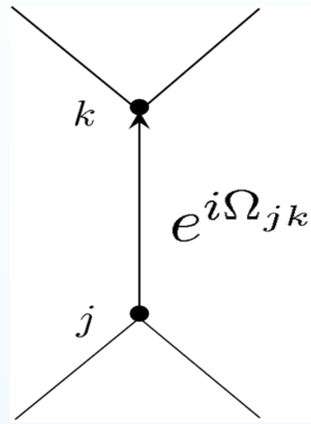

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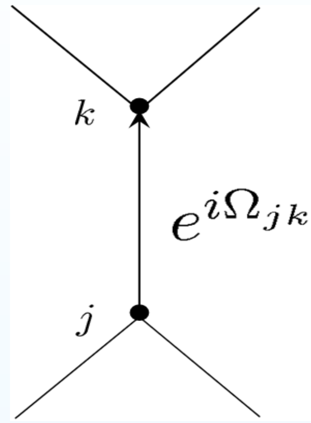

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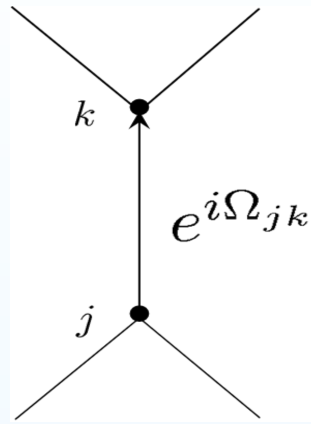

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$\Omega \mapsto H(\Omega)$, Hamiltonians parameterised by rep'n of $H_1(G)$.

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Regard as combinatorial graph. . .

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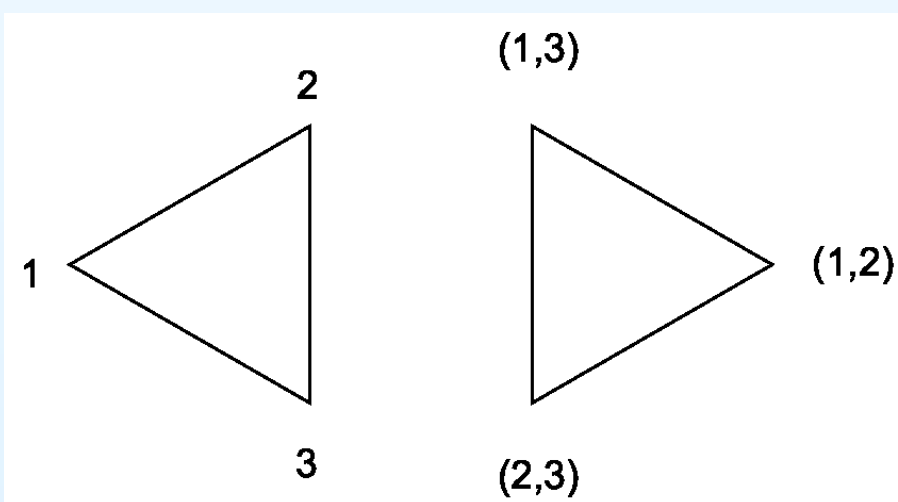

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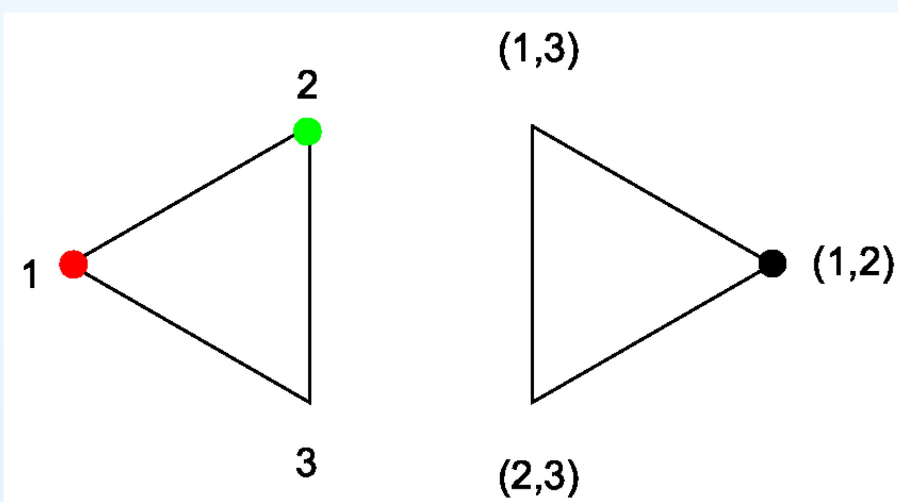

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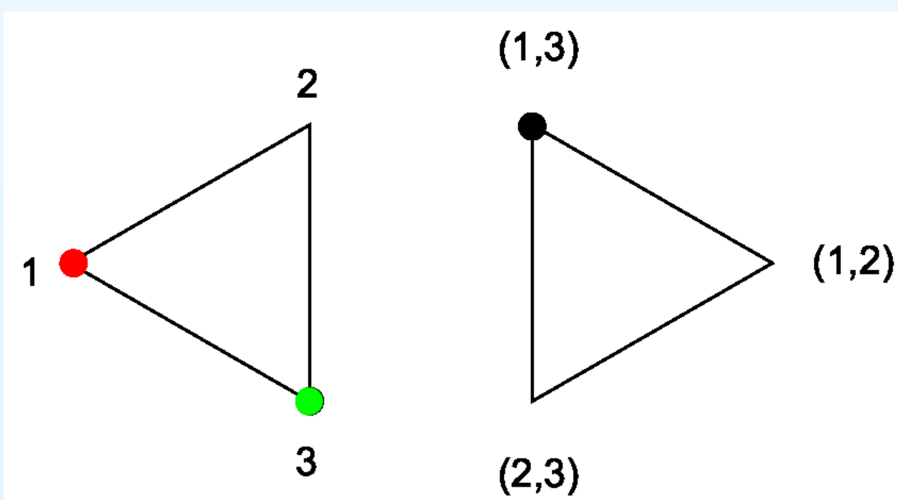

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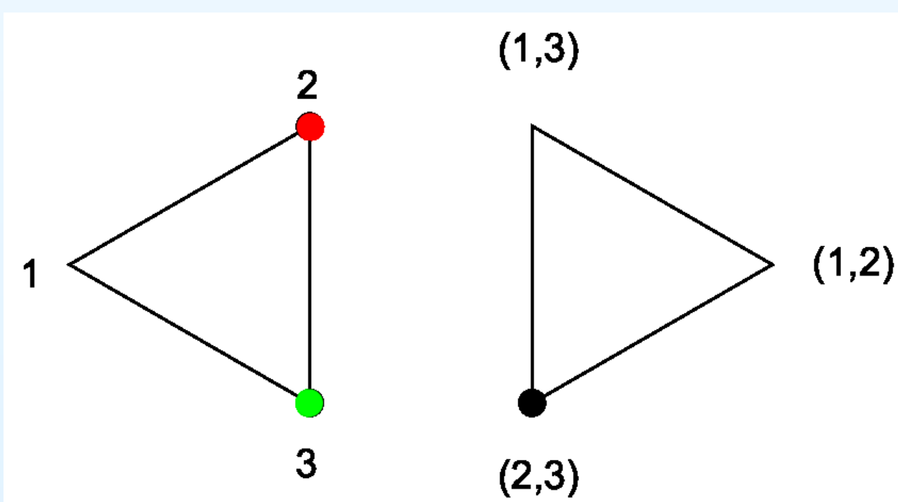
$$G_n = C_n(G) \;\; = \;\; \{V^n - \Delta_n\}/S_n$$

Regard as combinatorial graph. . .

Edges on G_n correspond to moving one particle along an edge of G while keeping the others fixed.

Examples ($n = 2$)

K_3



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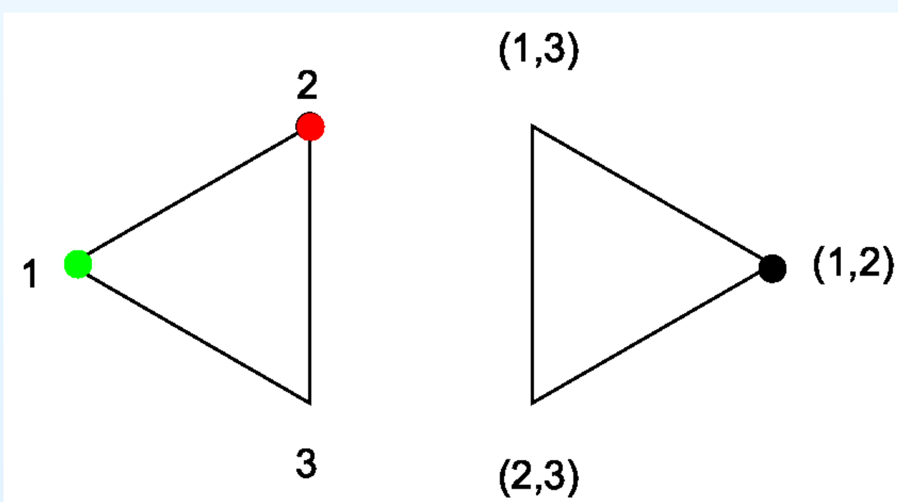

A c_2 cycle – two particles exchanged around a cycle.

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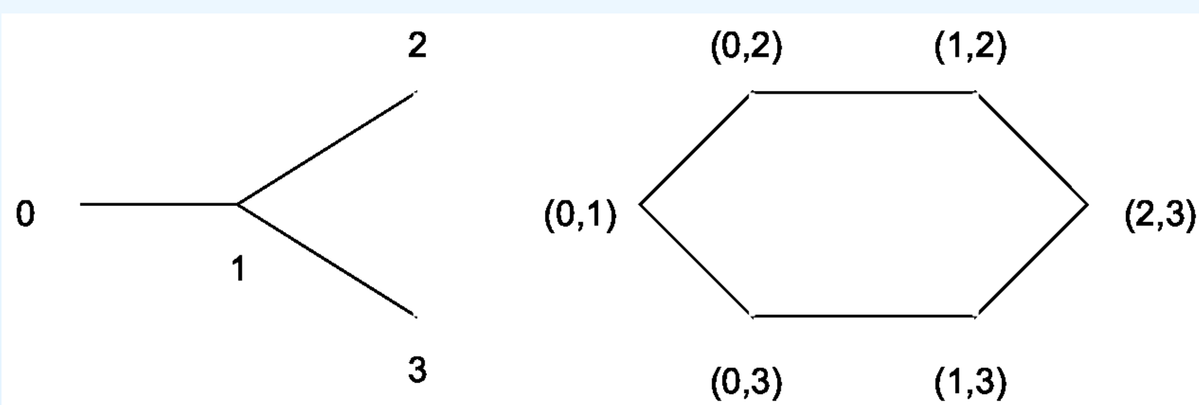

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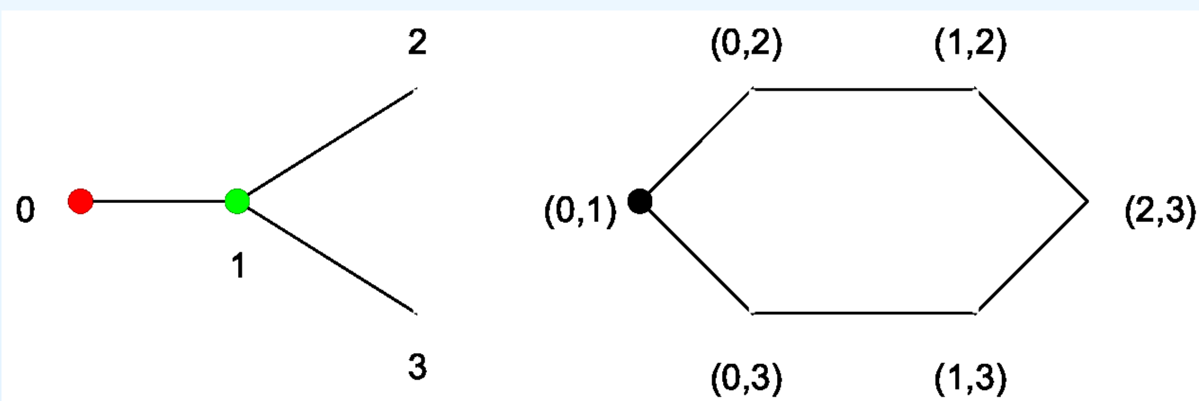

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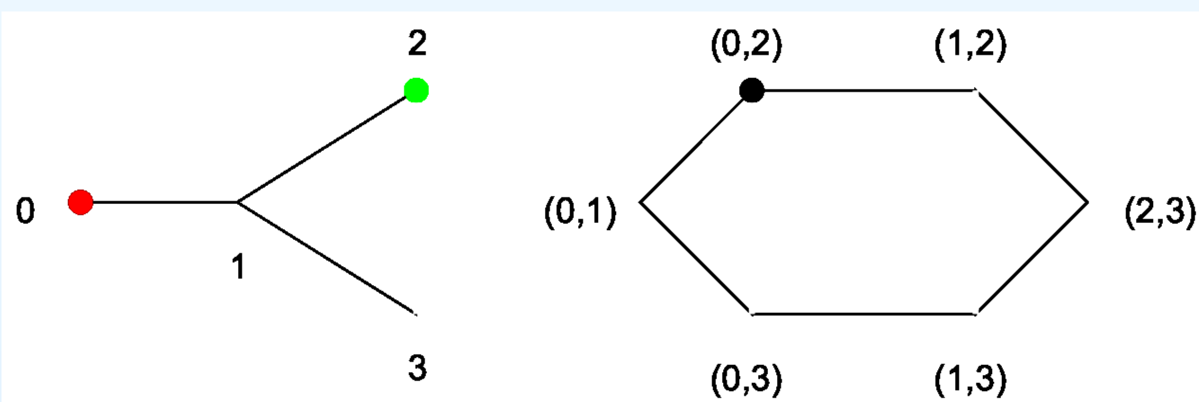

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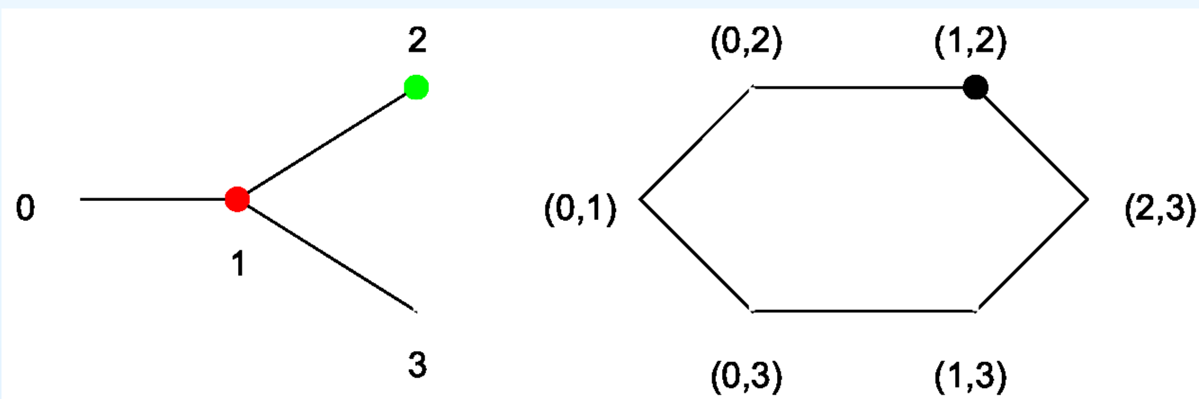

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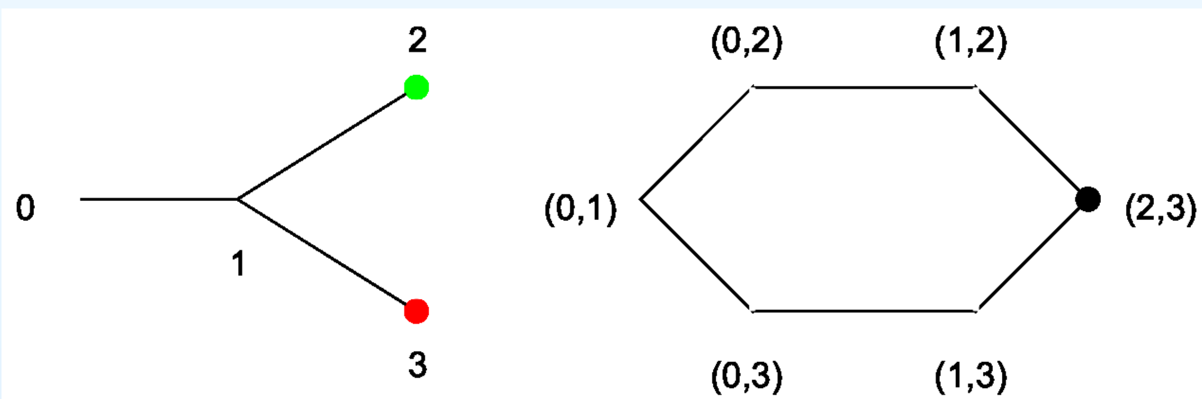

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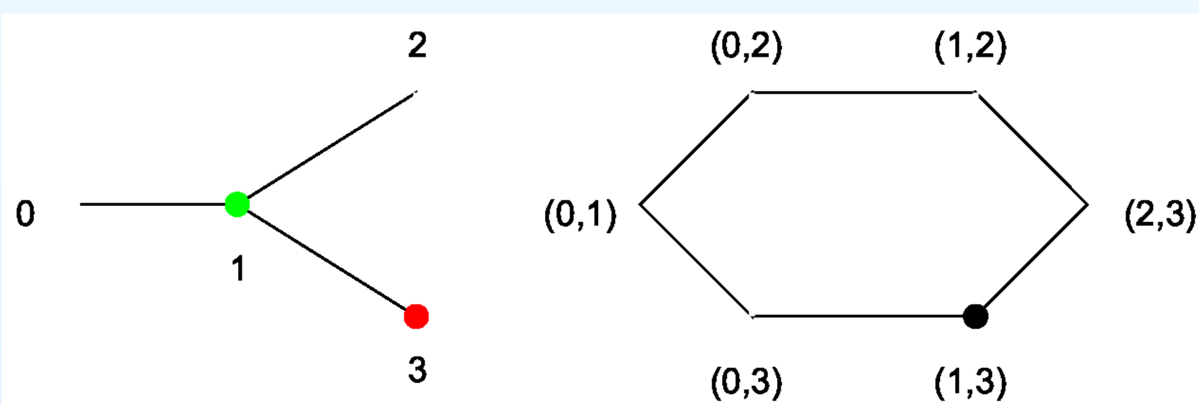

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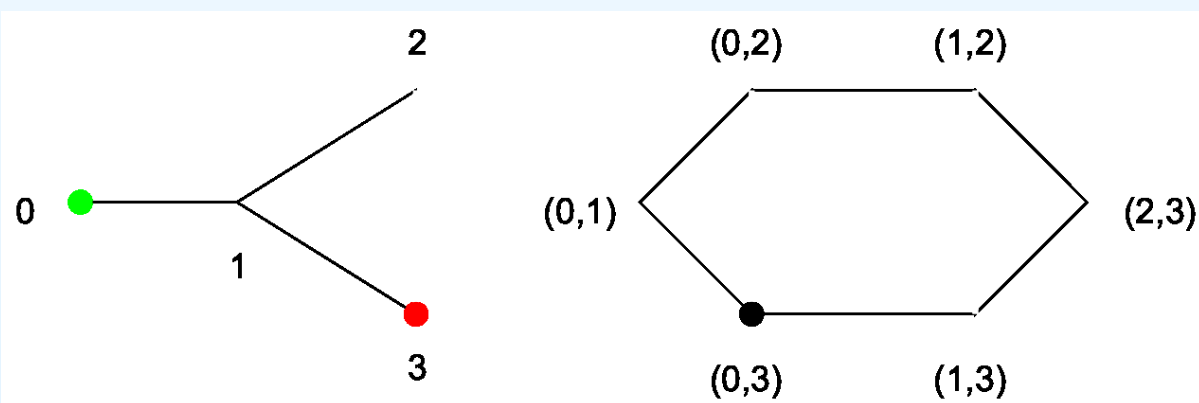

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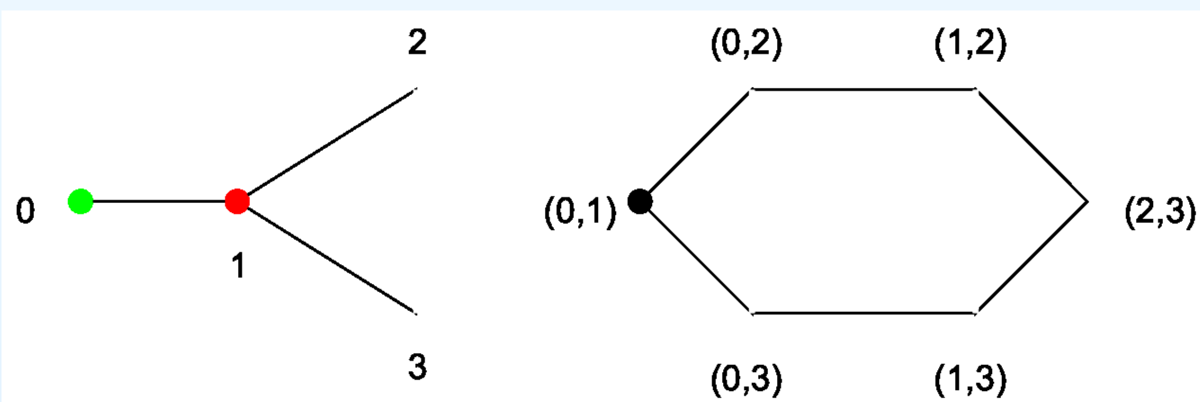

A Y-cycle – two particles exchanged on a Y junction.

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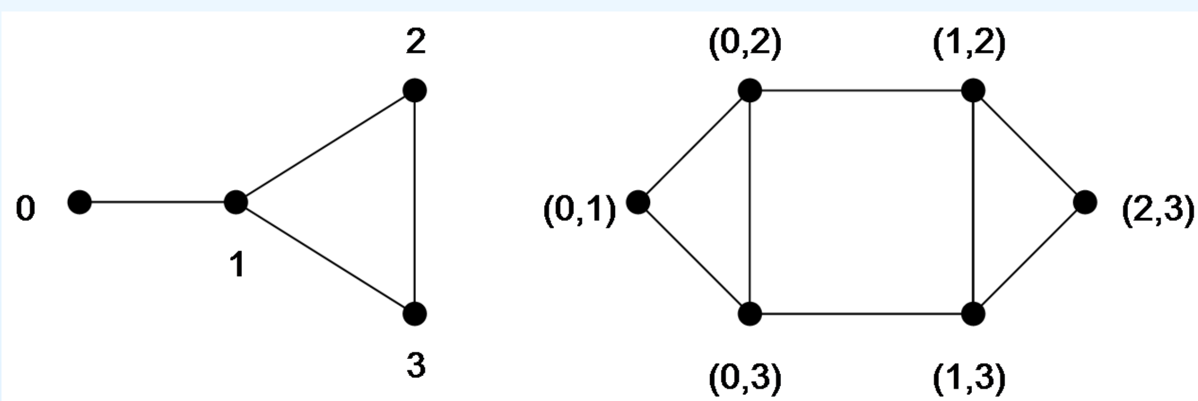

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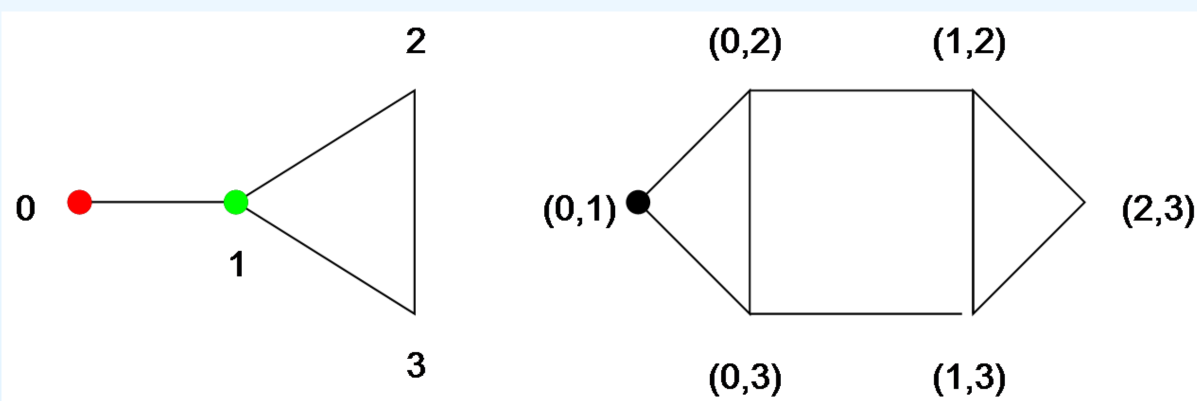

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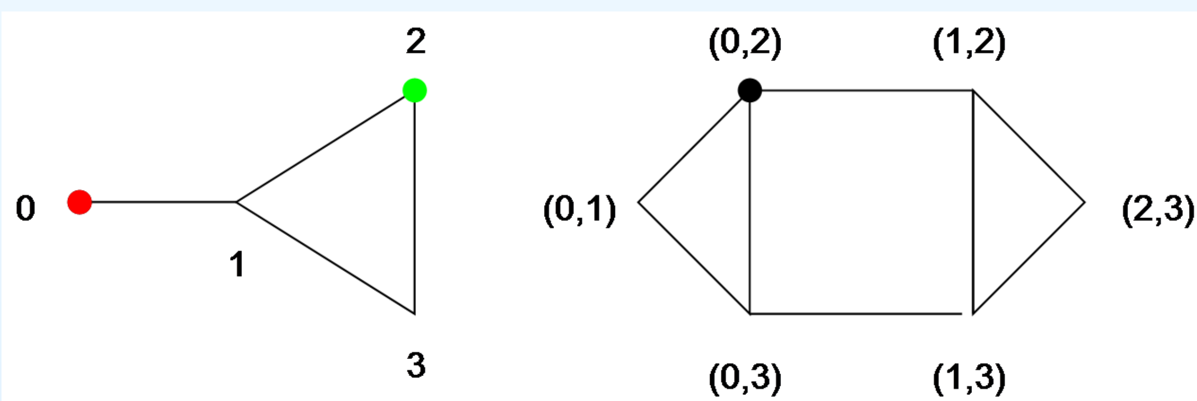

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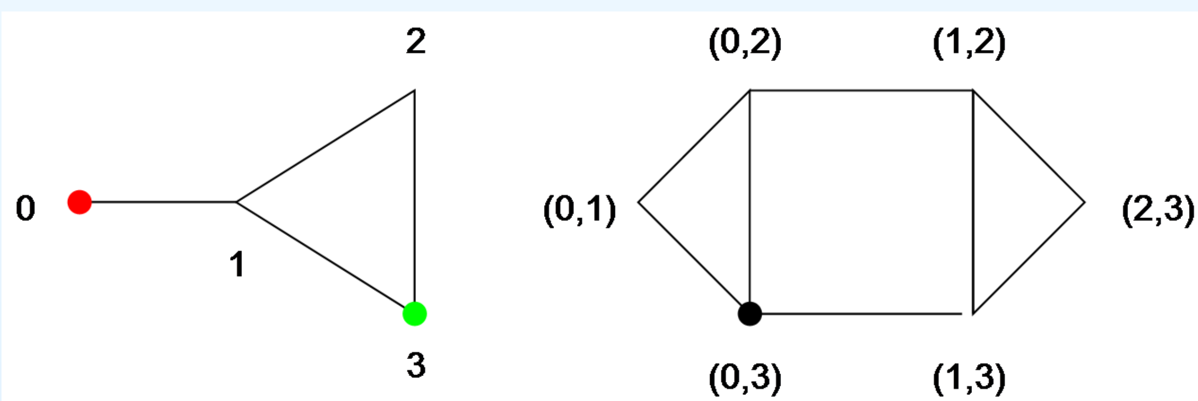

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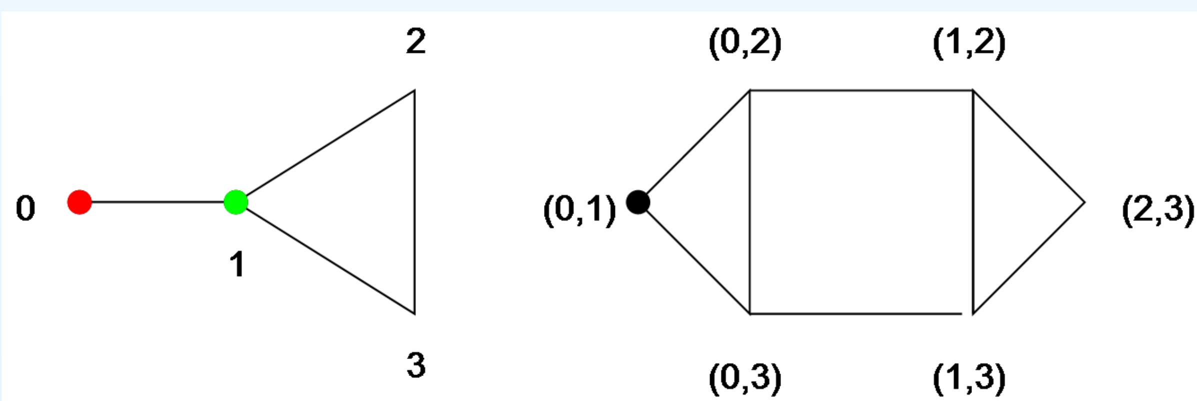

An AB-cycle – one particle goes around a cycle

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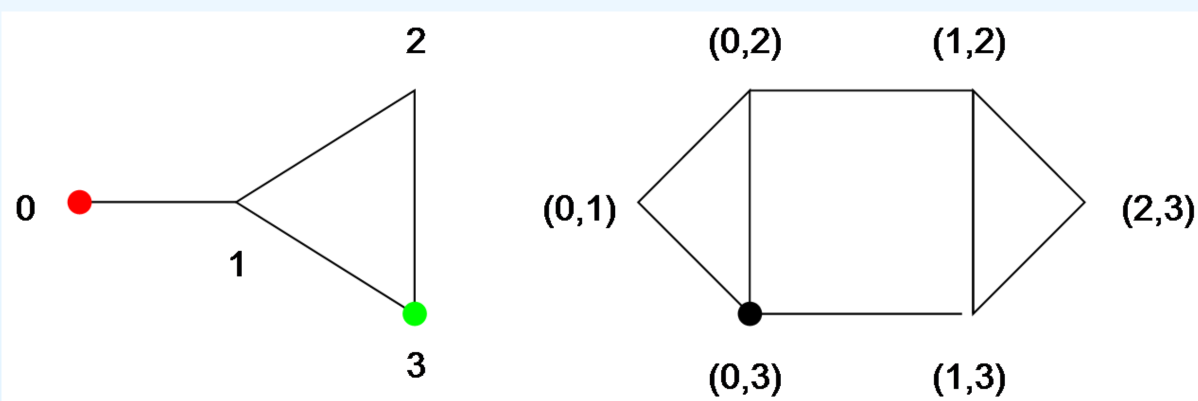

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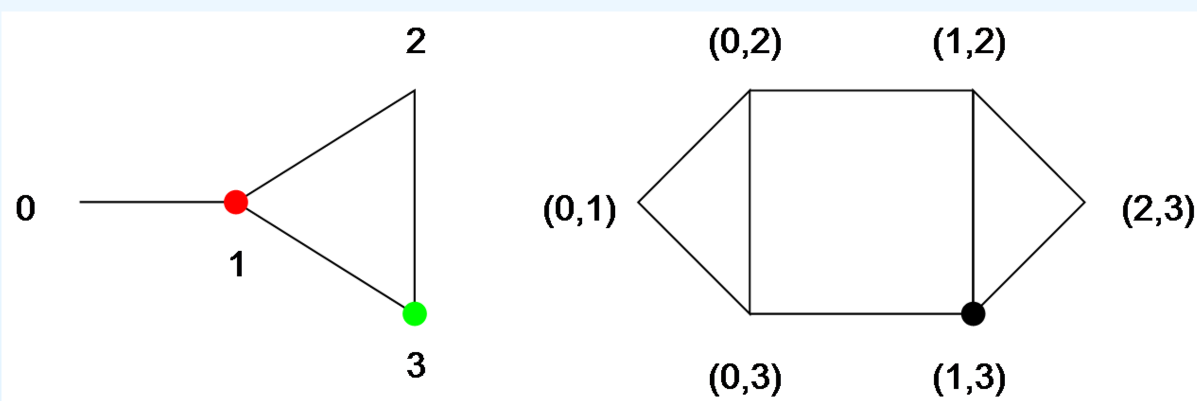

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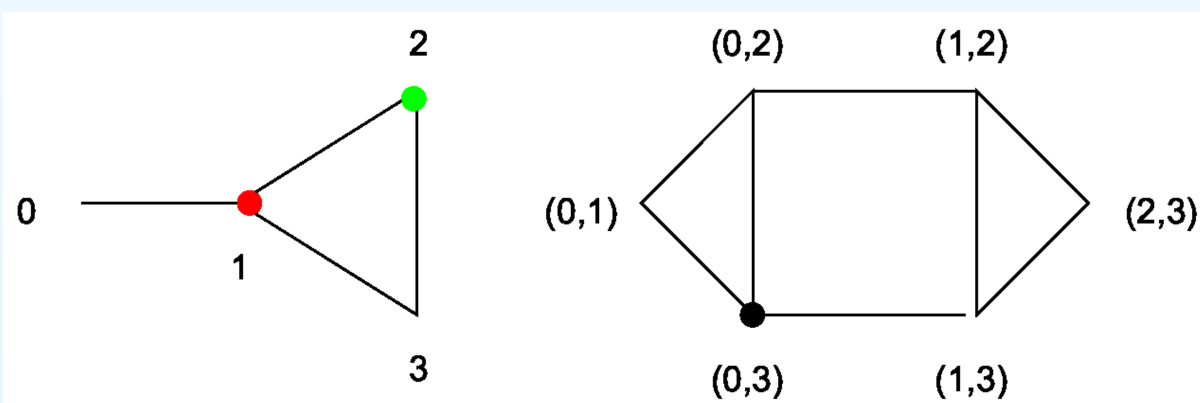

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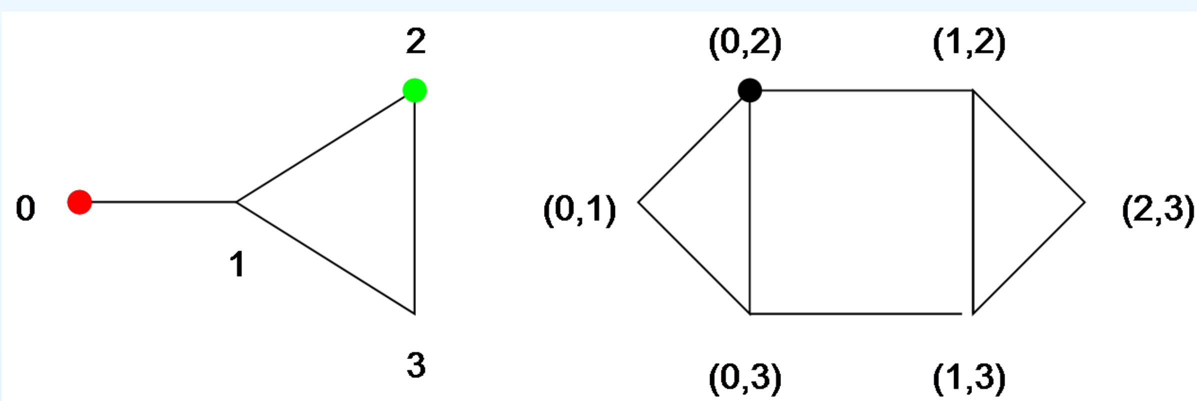

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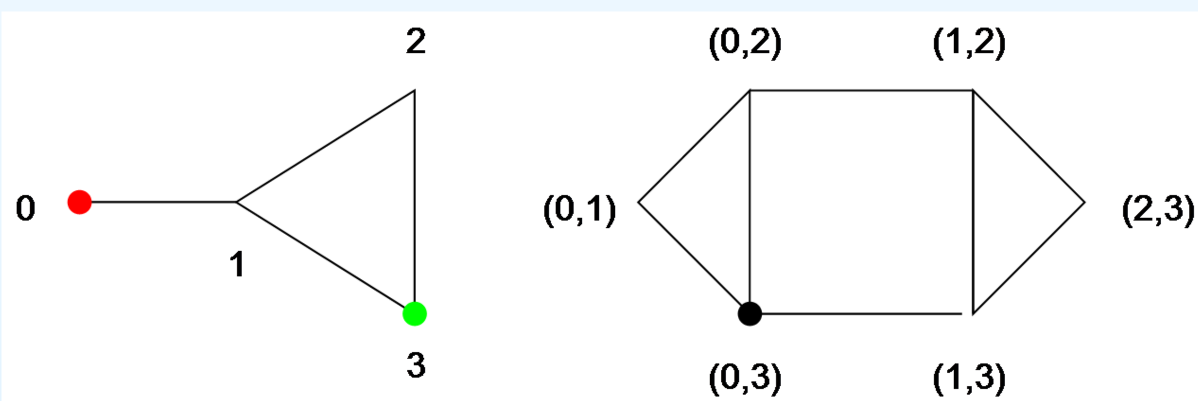
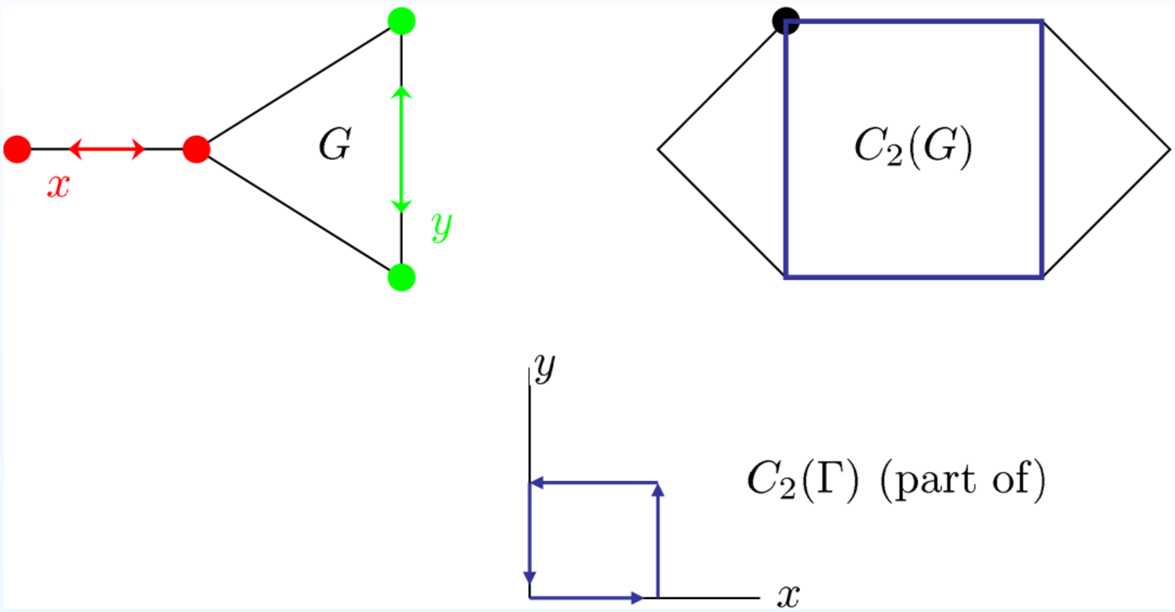


This is an example of a contractible cycle.

Contractible cycles

We will contractible cycles on G_n as trivial.



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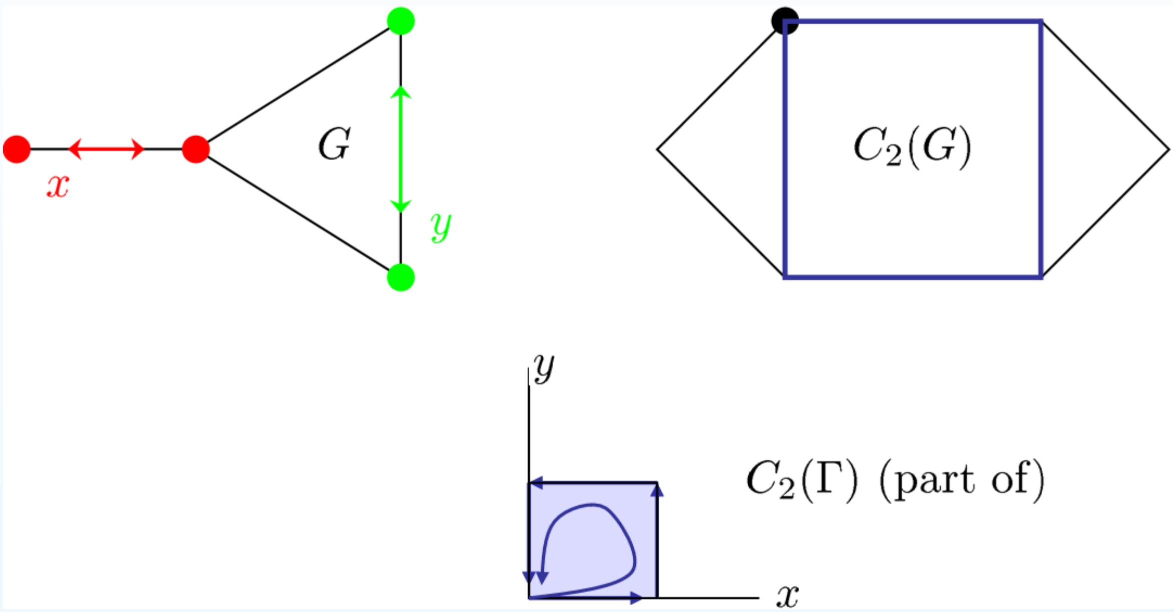

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If G is *sufficiently subdivided*, then

$$\pi_1^c(G_n)/T^c(G_n) \cong \pi_1(\Gamma_n)$$

Abrams (2000)

Sufficiently subdivided: Every path in G between vertices of degree not equal to two passes through at least n edges, and every cycle in G contains at least $n+1$ edges (can always be achieved by adding vertices to subdivide edges)

Topological gauge potentials

Abelian statistics on G_n determined by a gauge potential $(\Omega_n)_{JK}$, where

$$\Omega_n(c) = 0 \mod 2\pi \text{ for every contractible cycle}$$

We'll call these topological gauge potentials.

Correspond to a 1d rep'n of $\pi_1(\Gamma_n)$,

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Given n-particle Hamiltonian H_n, e.g. sum of 1-particle discrete Laplacians, abelian statistics is incorporated via

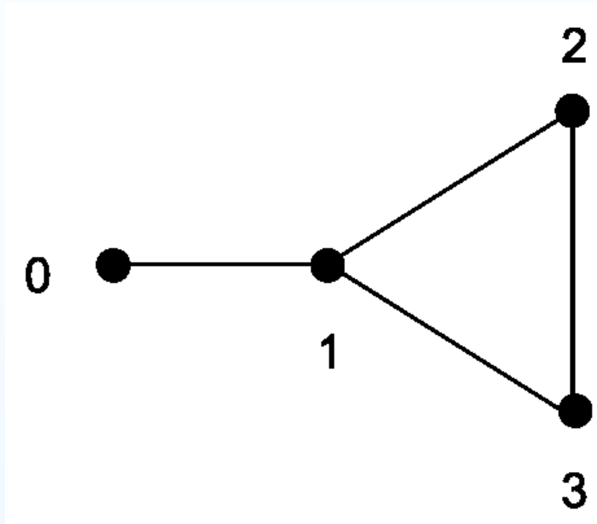
$$H_n(\Omega)_{JK} = \exp(i(\Omega_n)_{JK})(H_n)_{JK}$$

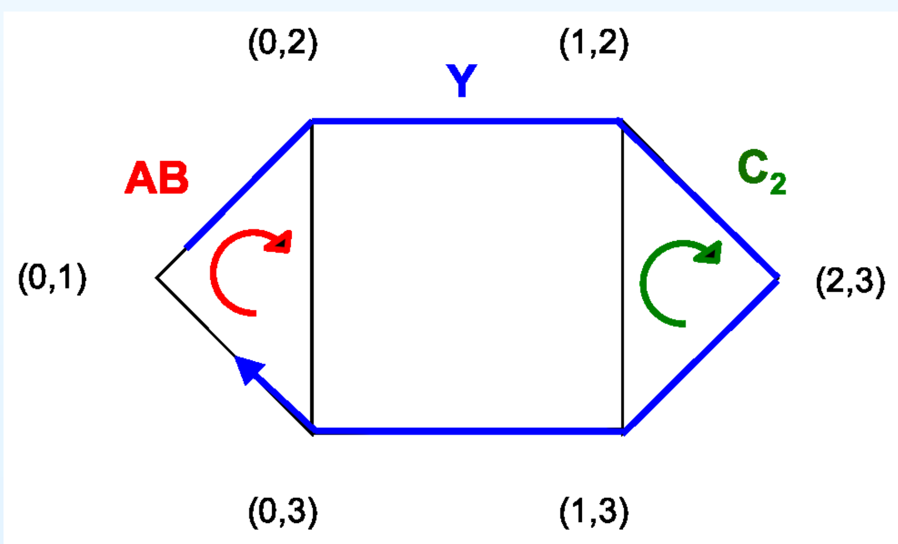
Problem: Calculate $H_1(\Gamma_n)$ in terms of graph invariants. . .

Ko and Park 2012 (discrete Morse theory)

Harrison, Keating, JR and Sawicki 2013

Key relation





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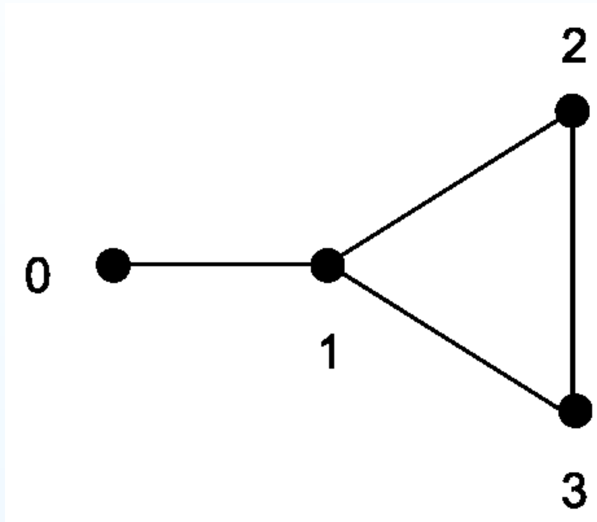

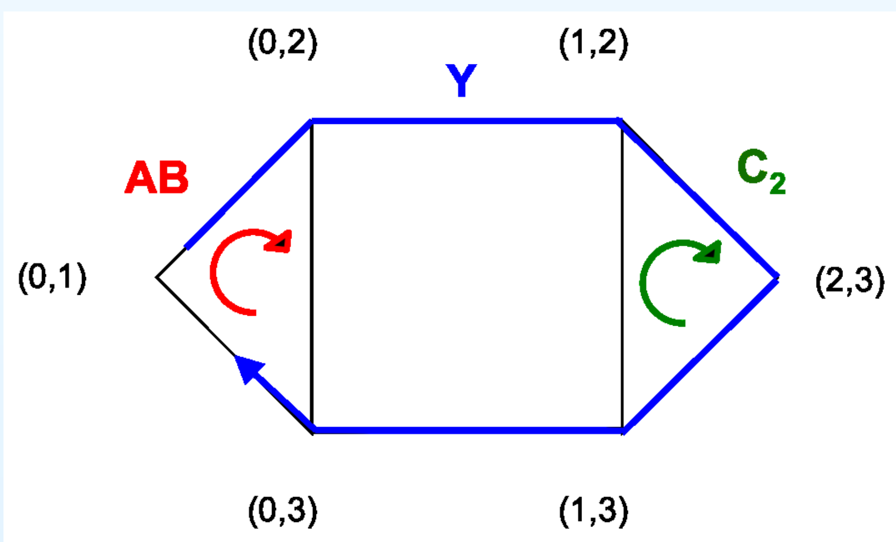

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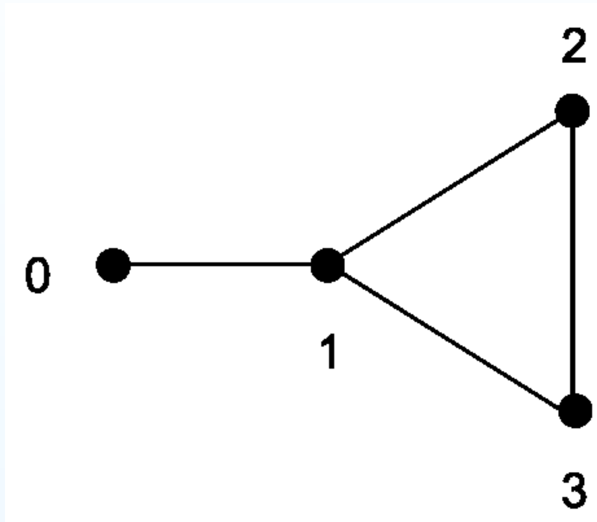

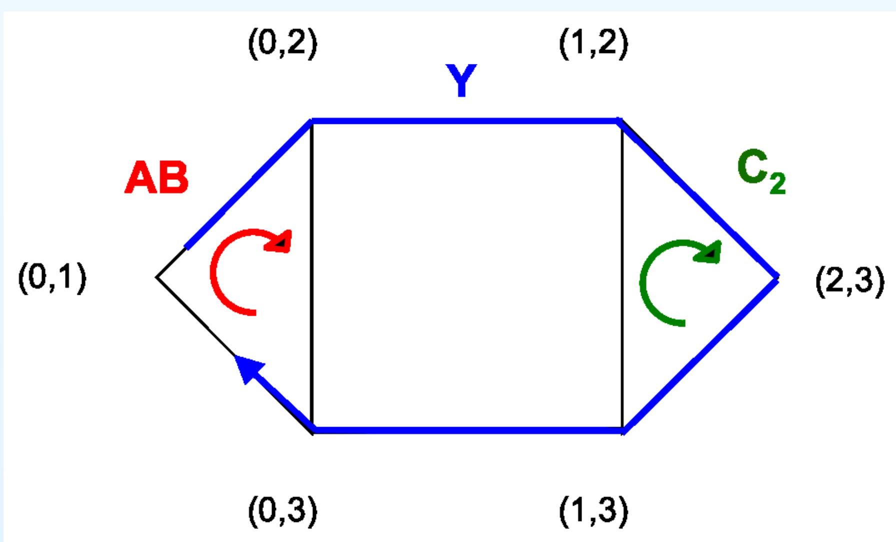

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Y's, AB's, and c_2's span $H_1(G_n) \ldots$

Connectivity

A graph is k-connected if it cannot be disconnected by removing $k - 1$ vertices.

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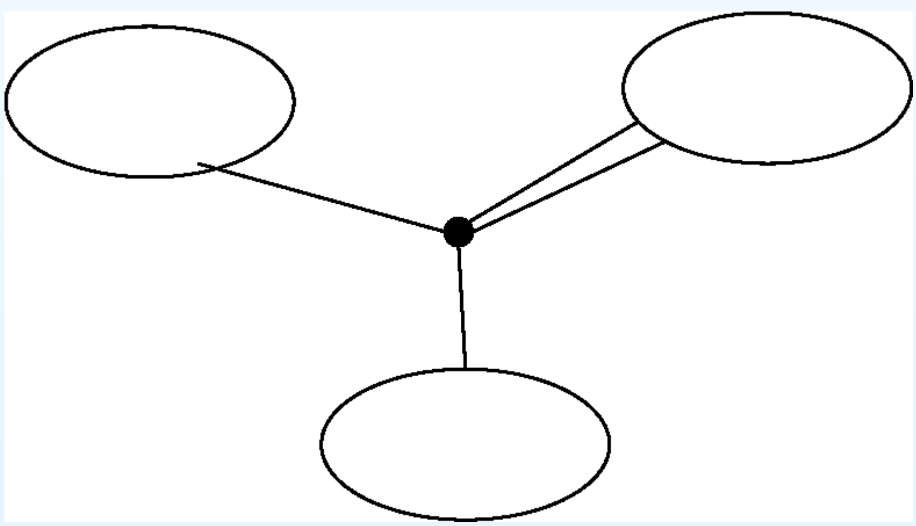
1-connected, not 2-connected



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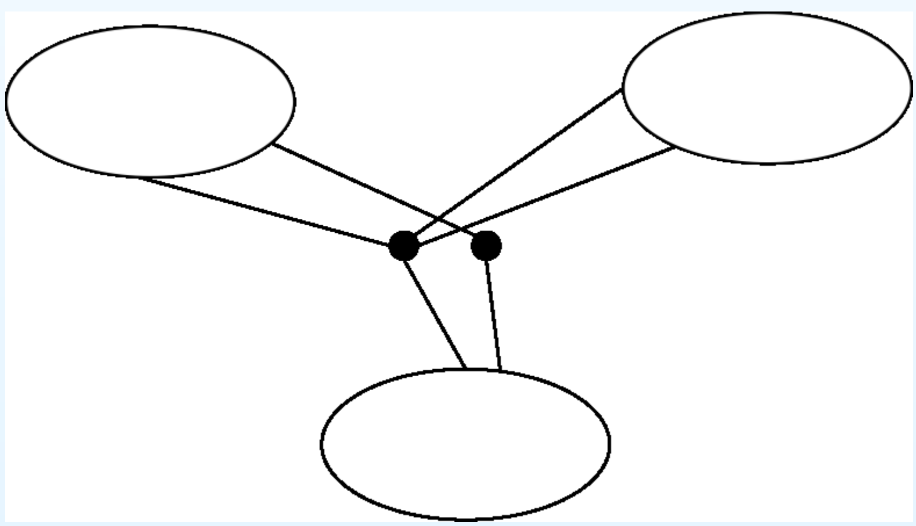

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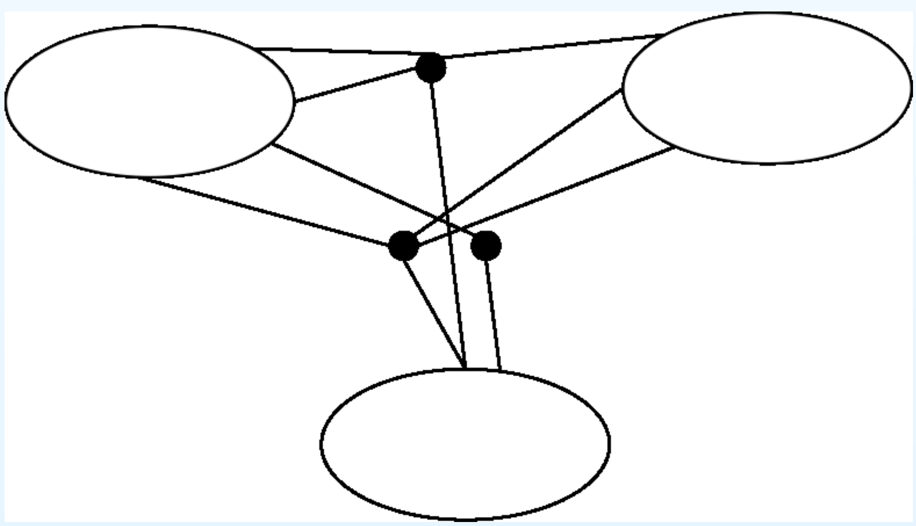
3-connected, not 4-connected



3-connected graphs

$$H_1(G_n) = \mathbb{Z}^\beta \oplus \begin{cases} \mathbb{Z}, & \text{if } G \text{ is planar,} \\ \mathbb{Z}/2, & \text{if } G \text{ is nonplanar.} \end{cases}$$

(follows from key relation and structure theorems for 3-connected graphs)

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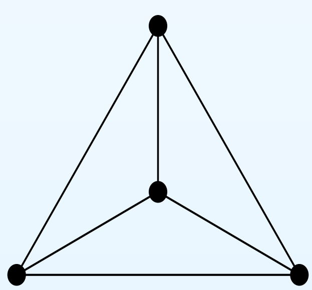
3-connected planar graphs: β AB phases and 1 anyon phase (like $\mathbb{R}^2$).



K_4: 3 AB phases, 1 anyon phase

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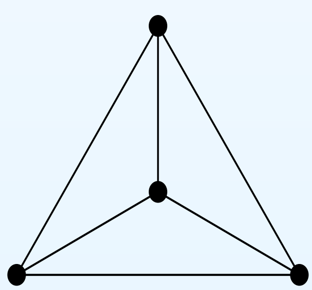
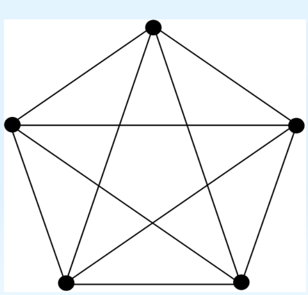


K_4: 3 AB phases, 1 anyon phase

3-connected nonplanar graphs: β AB phases and 1 sign (like $\mathbb{R}^{3+1}$).



K_5: 6 AB phases, 1 sign

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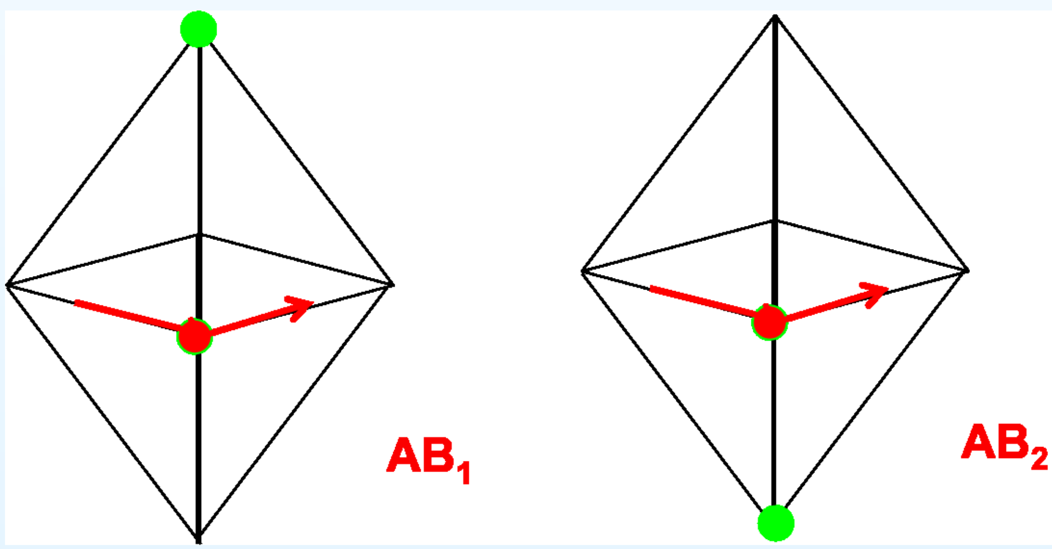

$\Omega(AB_1) \neq \Omega(AB_2)$, in general

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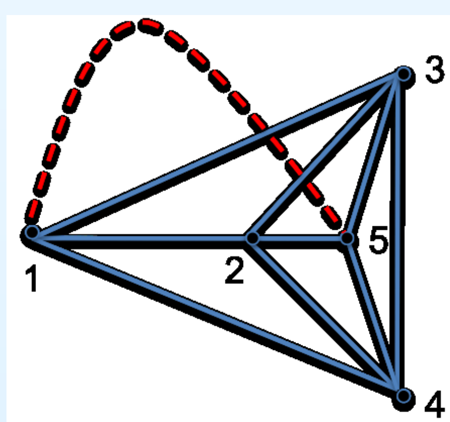
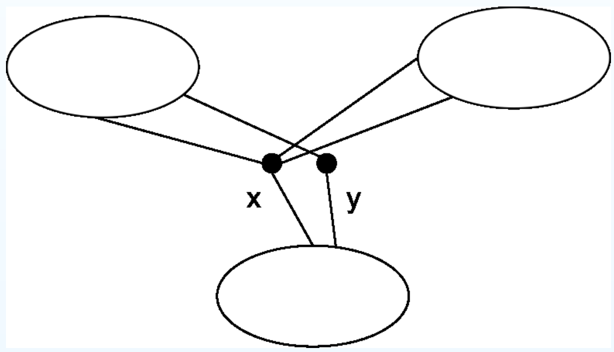
A puzzle, perhaps. . .



2-connected graphs

G may be decomposed into 3-connected components and cycles. . .



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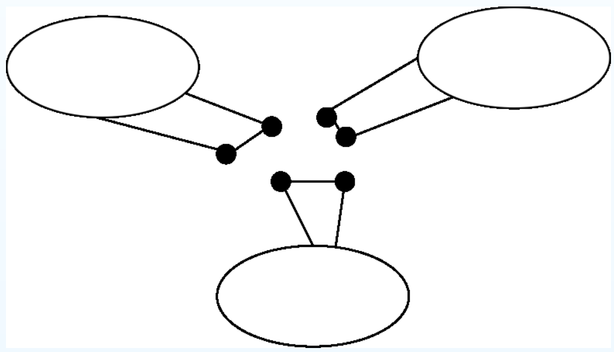

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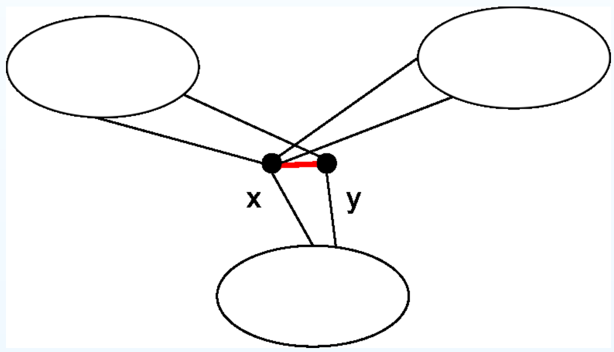

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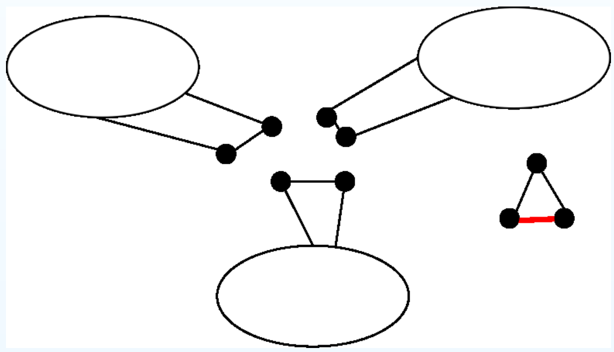

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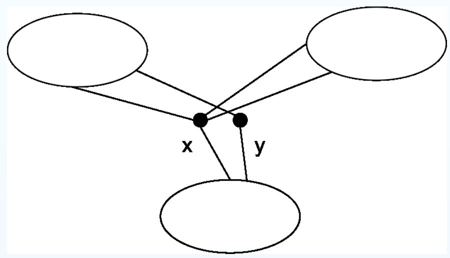

$\mu(x_i, y_i)$, # of connected components at two-vertex cut x_i, y_i

$N_2 = \sum_i \frac{1}{2}(\mu(x_i, y_i) - 1)(\mu(x_i, y_i) - 2)$

N_3, # of planar 3-connected components

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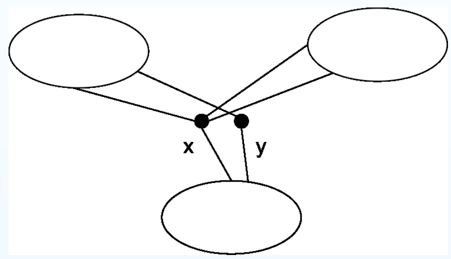

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$$H_1(G_n) = \mathbb{Z}^{\beta + N_2 + N_3} \oplus (\mathbb{Z}/2)^{N_3'}$$

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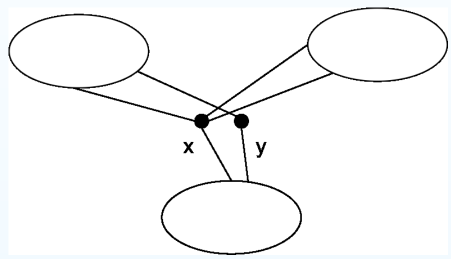

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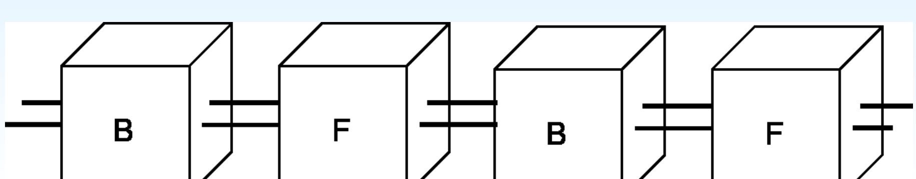
N_3, # of planar 3-connected components

N_3', # of nonplanar 3-connected components

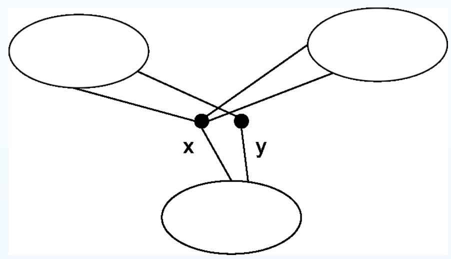
$$H_1(G_n) = \mathbb{Z}^{\beta + N_2 + N_3} \oplus (\mathbb{Z}/2)^{N_3'}$$

Chain of 3-connected components



2-connected graphs

G may be decomposed into 3-connected components and cycles. . .



$\mu(x_i, y_i)$, # of connected components at two-vertex cut x_i, y_i

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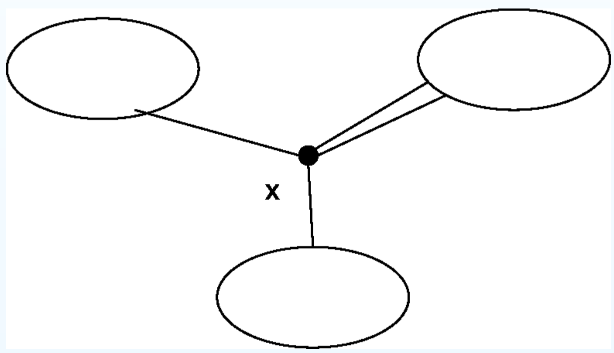
$$H_1(G_n) = \mathbb{Z}^{\beta + N_2 + N_3} \oplus (\mathbb{Z}/2)^{N_3'}$$

Building-up principle: $H_1(G_n)$ is independent of n for $n \geq 2$. Prescription for n-particle gauge potential in terms of 2-particle gauge potential.

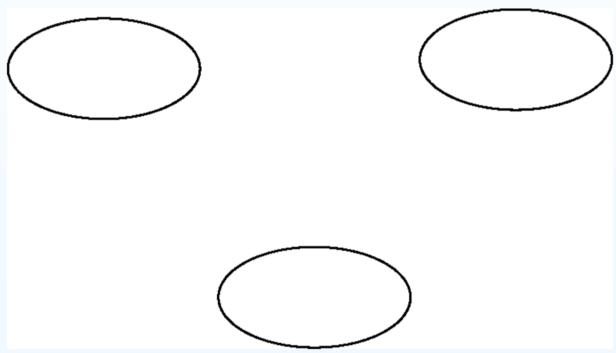
1-connected graphs

G may be decomposed into 2-connected components. . .



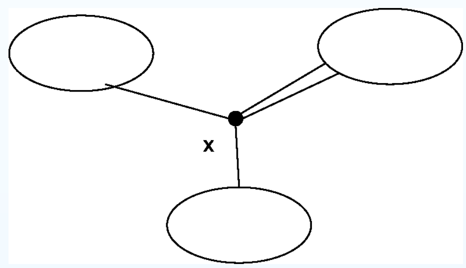
1-connected graphs

G may be decomposed into 2-connected components. . .



1-connected graphs

G may be decomposed into 2-connected components. . .



μ, # of connected components at one-vertex cut x_i

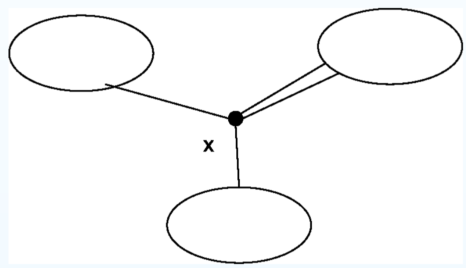
ν, # of edges at one-vertex cut x_i

$N_1(x_i) = \binom{n+\mu-2}{\mu-1}(\nu - 2) - \binom{n+\mu-2}{\mu-2} - (\nu - \mu - 1)$

$N_1 = \sum_i N_1(x_i)$

1-connected graphs

G may be decomposed into 2-connected components. . .



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$N_1(x_i) = \binom{n+\mu-2}{\mu-1}(\nu-2) - \binom{n+\mu-2}{\mu-2} - (\nu-\mu-1)$

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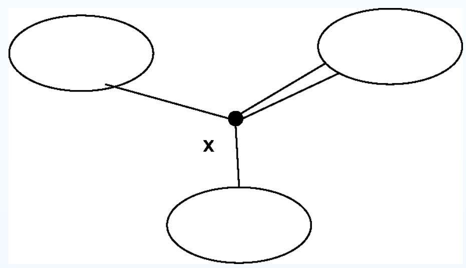
$$H_1(G_n) = \mathbb{Z}^{\beta+N_1+N_2+N_3} \oplus (\mathbb{Z}/2)^{N_3'}$$

Depends on number of particles.

1-connected graphs

G may be decomposed into 2-connected components. . .



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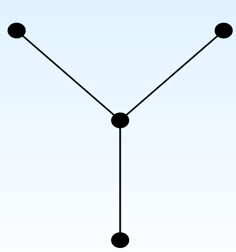
$$H_1(G_n) = \mathbb{Z}^{\beta+N_1+N_2+N_3} \oplus (\mathbb{Z}/2)^{N_3'}$$

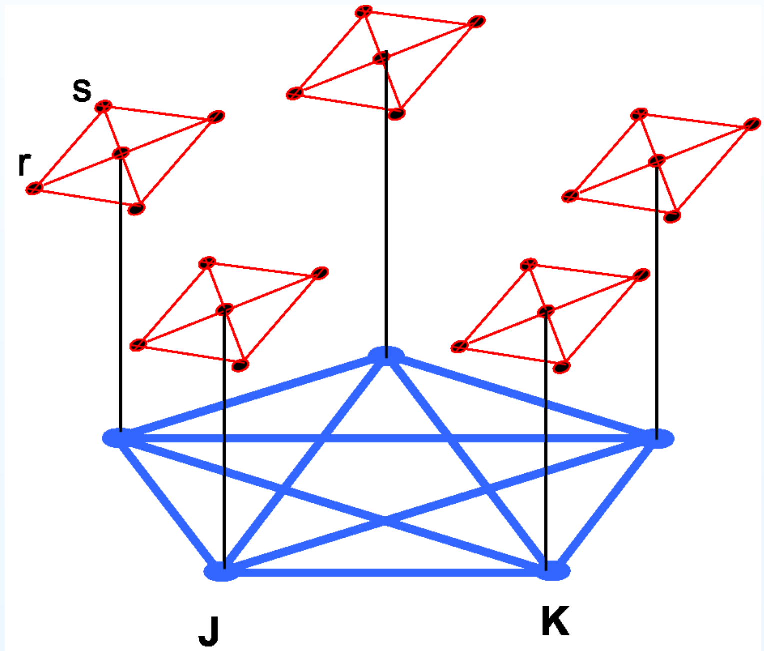
Depends on number of particles.



n-particle Y graph: $\binom{n}{2}$ phases

How phases might arise

$\mathcal{H} = \mathbb{C}^d$ (fast) $\times\, \mathbb{C}^D$ (slow)



$$H_{Jr,Ks} = h_{rs}(J)\delta_{JK} + \epsilon H_{JK}\delta_{rs}$$

Adiabatic approximation introduces gauge potential in slow Hamiltonian . . .

$$\Omega_{JK} \sim \text{Im}\,\langle v(J)|v(K)\rangle$$

Can regard J, r as multiparticle indices. Ω_{JK} does not automatically satisfy the topological condition.

An engineer's questions

- physical effects
- physical models

Happy Birthday, Yosi!