Spectral Shift Functions and some of their Applications in Mathematical Physics

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based on collaborations with

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A short course on the spectral shift function ξ

Notation:

$\mathcal{B}(\mathcal{H})$ denotes the Banach space of all **bounded** operators on $\mathcal{H}$ (a complex, separable Hilbert space).

$\mathcal{B}_\infty(\mathcal{H})$ denotes the Banach space of all **compact** operators on $\mathcal{H}$.

$\mathcal{B}_s(\mathcal{H})$, $s \geq 1$, denote the usual **Schatten–von Neumann trace ideals**.

Given two **self-adjoint** operators H, H_0 in $\mathcal{H}$, we assume one of the following (we denote $V = [H - H_0]$ whenever $\mathrm{dom}(H) = \mathrm{dom}(H_0)$):

- **Trace class assumption:** $V = [H - H_0] \in \mathcal{B}_1(\mathcal{H})$.
- **Relative trace class:** $V(H_0 - zI_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H})$.
- **Resolvent comparable:** $\big[(H - zI_{\mathcal{H}})^{-1} - (H_0 - zI_{\mathcal{H}})^{-1}\big] \in \mathcal{B}_1(\mathcal{H})$.

A short course on the spectral shift function ξ

The **Krein–Lifshitz spectral shift function** $\xi(\lambda; H, H_0)$ is a real-valued function on $\mathbb{R}$ that satisfies the **trace formula**: For appropriate functions f,

$$\boxed{\operatorname{tr}_{\mathcal{H}}(f(H) - f(H_0)) = \int_{\mathbb{R}} f'(\lambda)\, \xi(\lambda; H, H_0)\, d\lambda.}$$

For example, **resolvents**,

$$\operatorname{tr}_{\mathcal{H}}\left[(H - zI_{\mathcal{H}})^{-1} - (H_0 - zI_{\mathcal{H}})^{-1}\right] = -\int_{\mathbb{R}} \frac{\xi(\lambda; H, H_0)\, d\lambda}{(\lambda - z)^2},$$

semigroups (if H, H_0 are bounded from below), etc.

E.g., for $f \in C^1(\mathbb{R})$ with $f'(\lambda) = \int_{\mathbb{R}} e^{-i\lambda s} d\sigma(s)$ and $d\sigma$ a finite signed measure; $f(\lambda) = (\lambda - z)^{-1}$; or $\widehat{f} \in L^1(\mathbb{R}; (1 + |p|)dp)$ (implies $f \in C^1(\mathbb{R})$), etc.

V. Peller has **necessary** conditions on f, and also **sufficient** conditions on f in terms of certain **Besov** spaces. These spaces are not that far apart ....

A short course on the spectral shift function ξ

Generally, $\xi(\lambda; H, H_0) = \operatorname{tr}_{\mathcal{H}}(E_{H_0}(\lambda) - E_H(\lambda))$ is **not** correct (if $\dim(\mathcal{H}) = \infty$)!
$[E_{H_0}(\lambda) - E_H(\lambda)]$ is **NOT** necessarily of trace class even for $V = H - H_0$ of rank one!

Recall the **Trace** and the **Determinant:** Let $\lambda_j(T)$, $j \in J$ ($J \subseteq \mathbb{N}$ an index set) denote the eigenvalues of $T \in \mathcal{B}_1(\mathcal{H})$, **counting algebraic multiplicity**. Then $\sum_{j\in J} \lambda_j\big((T^*T)^{1/2}\big) < \infty$ and

$$\operatorname{tr}_{\mathcal{H}}(T) = \sum_{j\in J} \lambda_j(K), \qquad \det_{\mathcal{H}}(I_{\mathcal{H}} - T) = \prod_{j\in J} [1 - \lambda_j(T)].$$

The **Perturbation Determinant:** H and H_0 self-adjoint in $\mathcal{H}$, $\;H = H_0 + V$,

$$D_{H/H_0}(z) = \det_{\mathcal{H}}\big((H - zI_{\mathcal{H}})(H_0 - zI_{\mathcal{H}})^{-1}\big) = \det_{\mathcal{H}}\big(I_{\mathcal{H}} + V(H_0 - zI_{\mathcal{H}})^{-1}\big).$$

Example. If $H_0 = -(d^2/dx^2)$, $\;H = -(d^2/dx^2) + V(\cdot)$ in $L^2(\mathbb{R}; dx)$, $V \in L^1(\mathbb{R}; (1 + |x|)dx)$, real-valued, then

$$D_{H/H_0}(z) = \textbf{Jost} \text{ function} = \textbf{Evans} \text{ function}.$$

A short course on the spectral shift function ξ

General **Krein's formula** for the trace class $V = [H - H_0] \in \mathcal{B}_1(\mathcal{H})$:

$$\xi(\lambda; H, H_0) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \operatorname{Im}(\ln(D_{H/H_0}(\lambda + i\varepsilon))) \text{ for a.e. } \lambda \in \mathbb{R}, \qquad (*)$$

where

$$D_{H/H_0}(z) = \det_{\mathcal{H}}\big((H - zI_{\mathcal{H}})(H_0 - zI_{\mathcal{H}})^{-1}\big) = \det_{\mathcal{H}}\big(I_{\mathcal{H}} + V(H_0 - zI_{\mathcal{H}})^{-1}\big).$$

Then

$$\int_{\mathbb{R}} |\xi(\lambda; H, H_0)|\, d\lambda \leqslant \|H - H_0\|_{\mathcal{B}_1(\mathcal{H})} \text{ and } \int_{\mathbb{R}} \xi(\lambda; H, H_0)\, d\lambda = \operatorname{tr}(H - H_0).$$

If $\dim(\operatorname{ran}(E_{H_0}(a - \epsilon, b + \epsilon))) < \infty$, then

$$\xi(b - 0; H, H_0) - \xi(a + 0; H, H_0) = \dim(\operatorname{ran}(E_{H_0}(a, b))) - \dim(\operatorname{ran}(E_H(a, b))). \quad (**)$$

Note. Formulas $(*)$ and $(**)$ remain **valid** in the **relative trace class case**, where $V(H_0 - zI_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H})$. $\Longrightarrow$ **Eigenvalue counting function.**

A short course on the spectral shift function ξ

Away from $\sigma_{ess}(H_0) = \sigma_{ess}(H)$, $\xi(\lambda; H, H_0)$ is **piecewise constant:**

Counting multiplicity: Suppose $\lambda_0 \in \mathbb{R}\backslash\sigma_{ess}(H_0)$. Then,

$$\xi(\lambda_0 + 0; H, H_0) - \xi(\lambda_0 - 0; H, H_0) = m(H_0; \lambda_0) - m(H; \lambda_0),$$

where $m(T, \lambda) \in \mathbb{N} \cup \{0\}$ denotes the multiplicity of $\lambda \in \sigma_p(T)$, $T = T^*$.

Scattering theory: The Birman–Krein formula

Let $S(H, H_0)$ be the scattering operator for the pair (H, H_0). Then

$$S(H, H_0) = \int_{\sigma_{ac}(H_0)}^{\oplus} S(\lambda; H, H_0)\, d\lambda \ \text{ in } \ \mathcal{H} \simeq \int_{\sigma_{ac}(H_0)}^{\oplus} \mathcal{K}\, d\lambda,$$

where $S(\lambda; H, H_0)$ is the fixed energy scattering operator in $\mathcal{K}$ (assuming **uniform spectral multiplicity of $\sigma_{ac}(H_0)$**, for simplicity). Then

$$\xi(\lambda; H, H_0) = -\frac{1}{2\pi i} \ln(\det_{\mathcal{K}}(S(\lambda; H, H_0)) \ \text{ for a.e. } \ \lambda \in \sigma_{ac}(H_0).$$

Applications: Fredholm Indices

Computing **Fredholm indices:** Consider the **model operator**

$$\mathbf{D_A} = (d/dt) + \mathbf{A}, \quad \mathrm{dom}(\mathbf{D_A}) = \mathrm{dom}(d/dt) \cap \mathrm{dom}(\mathbf{A}_-) \ \text{ in } L^2(\mathbb{R}; \mathcal{H}),$$

where $\mathrm{dom}(d/dt) = W^{1,2}(\mathbb{R}; \mathcal{H})$, and

$$\mathbf{A} = \int_{\mathbb{R}}^{\oplus} A(t)\, dt, \quad \mathbf{A}_- = \int_{\mathbb{R}}^{\oplus} A_-\, dt \ \text{ in } L^2(\mathbb{R}; \mathcal{H}) \simeq \int_{\mathbb{R}}^{\oplus} \mathcal{H}\, dt,$$

$$A_\pm = \lim_{t \to \pm\infty} A(t) \ \text{ exist in norm resolvent sense and are boundedly invertible, i.e., } 0 \in \rho(A_\pm),$$

and where we consider the case of **relative trace class** perturbations $[A(t) - A_-]$,

$$A(t) = A_- + B(t), \quad t \in \mathbb{R},$$

$$B(t)(A_- - zI_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H}), \quad t \in \mathbb{R} \quad \text{(plus quite a bit more } \longrightarrow \text{ next page)},$$

such that $\mathbf{D_A}$ becomes a Fredholm operator in $L^2(\mathbb{R}; \mathcal{H})$.

Applications: Fredholm Indices

Just for clarity,

$$\mathbf{A} = \int_{\mathbb{R}}^{\oplus} A(t)\, dt \ \text{ in } L^2(\mathbb{R}; \mathcal{H}) \simeq \int_{\mathbb{R}}^{\oplus} \mathcal{H}\, dt, \quad \mathcal{H} \text{ a separable, complex } H\text{-space},$$

denotes the operator in $L^2(\mathbb{R}; \mathcal{H})$ defined by

$$(\mathbf{A}f)(t) = A(t)f(t) \ \text{ for a.e. } t \in \mathbb{R},$$

$$f \in \operatorname{dom}(\mathbf{A}) = \bigg\{ g \in L^2(\mathbb{R}; \mathcal{H}) \,\bigg|\, g(t) \in \operatorname{dom}(A(t)) \text{ for a.e. } t \in \mathbb{R},$$

$$t \mapsto A(t)g(t) \text{ is (weakly) measurable, } \int_{\mathbb{R}} \|A(t)g(t)\|_{\mathcal{H}}^2\, dt < \infty \bigg\}.$$

Applications: Fredholm Indices, contd.

Main Hypotheses (we're aiming at $A(t) = A_- + B(t)$ ....):

- A_- – **self-adjoint** on $\mathrm{dom}(A_-) \subseteq \mathcal{H}$, $\mathcal{H}$ a complex, separable Hilbert space.

- $B(t)$, $t \in \mathbb{R}$, – **closed, symmetric,** in $\mathcal{H}$, $\mathrm{dom}(B(t)) \supseteq \mathrm{dom}(A_-)$.

- There exists a family $B'(t)$, $t \in \mathbb{R}$, – **closed, symmetric,** in $\mathcal{H}$, with

  $\mathrm{dom}(B'(t)) \supseteq \mathrm{dom}(A_-)$, such that

  $B(t)(|A_-| + I_{\mathcal{H}})^{-1}$, $t \in \mathbb{R}$, is **weakly locally a.c.** and for a.e. $t \in \mathbb{R}$,

  $$\frac{d}{dt}\big(g, B(t)(|A_-| + I_{\mathcal{H}})^{-1}h\big)_{\mathcal{H}} = \big(g, B'(t)(|A_-| + I_{\mathcal{H}})^{-1}h\big)_{\mathcal{H}}, \quad g, h \in \mathcal{H}.$$

- $B'(\cdot)(|A_-| + I_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H})$, $t \in \mathbb{R}$, $\int_{\mathbb{R}} \big\|B'(t)(|A_-| + I_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}_1(\mathcal{H})}\, dt < \infty$.

- $\big\{(|B(t)|^2 + I_{\mathcal{H}})^{-1}\big\}_{t\in\mathbb{R}}$ and $\big\{(|B'(t)|^2 + I_{\mathcal{H}})^{-1}\big\}_{t\in\mathbb{R}}$ are **weakly measurable.**

Applications: Fredholm Indices, contd.

Consequences of these hypotheses:

$A(t) = A_- + B(t), \quad \operatorname{dom}(A(t)) = \operatorname{dom}(A_-), \quad t \in \mathbb{R}.$

There exists $A_+ = A(+\infty) = A_- + B(+\infty), \quad \operatorname{dom}(A_+) = \operatorname{dom}(A_-),$

$\text{n-lim}_{t\to\pm\infty}(A(t) - zI_{\mathcal{H}})^{-1} = (A_\pm - zI_{\mathcal{H}})^{-1},$

$(A_+ - A_-)(A_- - zI_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H}),$

$\big[(A(t) - zI_{\mathcal{H}})^{-1} - (A_\pm - zI_{\mathcal{H}})^{-1}\big] \in \mathcal{B}_1(\mathcal{H}), \quad t \in \mathbb{R},$

$\big[(A_+ - zI_{\mathcal{H}})^{-1} - (A_- - zI_{\mathcal{H}})^{-1}\big] \in \mathcal{B}_1(\mathcal{H}),$

$\sigma_{\rm ess}(A(t)) = \sigma_{\rm ess}(A_+) = \sigma_{\rm ess}(A_-), \quad t \in \mathbb{R}.$

Lemma.

Under these assumptions, and if $\boxed{0 \in \rho(A_+) \cap \rho(A_-),}$ $\longleftrightarrow$ **Fredholm Property**

then $\mathbf{D_A} = \frac{d}{dt} + \mathbf{A}$ on $\operatorname{dom}(\mathbf{D_A}) = \operatorname{dom}(d/dt) \cap \operatorname{dom}(\mathbf{A}_-)$, is **closed** and a **Fredholm** operator in $L^2(\mathbb{R}; \mathcal{H})$.

Applications: Fredholm Indices, contd.

The following is proved in

F.G., Y. Latushkin, K. Makarov, F. Sukochev, and Y. Tomilov, Adv. Math. **227**, 319–420 (2011):

Theorem.

Under these assumptions, and if $\boxed{0 \in \rho(A_+) \cap \rho(A_-),}$ $\longleftrightarrow$ **Fredholm Property**

$$\operatorname{ind}(\mathbf{D_A}) = \dim(\ker(\mathbf{D_A})) - \dim(\ker(\mathbf{D_A^*})) \qquad \textbf{Fredholm Index}$$

$$= \xi(0_+; \mathbf{H}_2, \mathbf{H}_1) \qquad \mathbf{H}_1 = \mathbf{D_A^*}\mathbf{D_A}, \ \mathbf{H}_2 = \mathbf{D_A}\mathbf{D_A^*}$$

$$= \operatorname{SpFlow}(\{A(t)\}_{t=-\infty}^{\infty}) \qquad \textbf{Spectral Flow}$$

$$= \xi(0; A_+, A_-) \quad \textbf{internal SSF} \qquad A_\pm = A(\pm\infty), \quad 0 \in \rho(A_\pm)$$

$$= \pi^{-1} \lim_{\varepsilon \downarrow 0} \operatorname{Im}\big(\ln\big(\det_{\mathcal{H}}((A_+ - i\varepsilon I_{\mathcal{H}})(A_- - i\varepsilon I_{\mathcal{H}})^{-1})\big)\big) \quad \textbf{Path Independence}$$

$$\mathbf{H}_1 = \mathbf{D_A^*}\mathbf{D_A} = -\frac{d^2}{dt^2} +_q \mathbf{V}_1, \ \mathbf{V}_1 = \mathbf{A}^2 - \mathbf{A}',$$

$$\mathbf{H}_2 = \mathbf{D_A}\mathbf{D_A^*} = -\frac{d^2}{dt^2} +_q \mathbf{V}_2, \ \mathbf{V}_2 = \mathbf{A}^2 + \mathbf{A}'.$$

"$+_q$" abbreviates the form sum, $\mathbf{T} = \int_{\mathbb{R}}^{\oplus} T(t)\, dt.$

Applications: Fredholm Indices, contd.

Two key elements in the proof:

Theorem (Trace Identity).

$$\operatorname{tr}_{L^2(\mathbb{R};\mathcal{H})}\left((\mathbf{H}_2 - z\,\mathbf{I})^{-1} - (\mathbf{H}_1 - z\,\mathbf{I})^{-1}\right) = \frac{1}{2z}\operatorname{tr}_{\mathcal{H}}(g_z(A_+) - g_z(A_-)),$$

where $g_z(x) = \frac{x}{\sqrt{x^2 - z}}$, $z \in \mathbb{C}\backslash[0,\infty)$, $x \in \mathbb{R}$, a smoothed-out sign fct.

Theorem (Pushnitski's Formula, an Abel-Type Transform).

$$\xi(\lambda; \mathbf{H}_2, \mathbf{H}_1) = \frac{1}{\pi}\int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)\,d\nu}{(\lambda - \nu^2)^{1/2}} \quad \text{for a.e. } \lambda > 0.$$

Relating the **external** SSF, $\xi(\,\cdot\,; \mathbf{H}_2, \mathbf{H}_1)$ and **internal** SSF, $\xi(\,\cdot\,; A_+, A_-)$:

Recalling, $\mathbf{D_A} = (d/dt) + \mathbf{A}$, $\mathbf{A} = \int_{\mathbb{R}}^{\oplus} A(t)\,dt$ in $L^2(\mathbb{R};\mathcal{H})$,

$\mathbf{H}_1 = \mathbf{D}_{\mathbf{A}}^*\mathbf{D}_{\mathbf{A}}$, $\mathbf{H}_2 = \mathbf{D}_{\mathbf{A}}\mathbf{D}_{\mathbf{A}}^*$, $A(t) \underset{t\pm\infty}{\longrightarrow} A_\pm$, etc.

Applications: Fredholm Indices, contd.

There's nothing special about **resolvent** differences of $\mathbf{H}_2$ and $\mathbf{H}_1$ on the l.h.s. of the trace identity!

Under appropriate conditions on f one obtains

$$\operatorname{tr}_{L^2(\mathbb{R};\mathcal{H})}\big(f(\mathbf{H}_2)-f(\mathbf{H}_1)\big)=\operatorname{tr}_{\mathcal{H}}\big(F(A_+)-F(A_-)\big),$$

where F is determined by f via

$$F(\nu)=-2^{-1}f(0)\operatorname{sgn}(\nu)+\frac{\nu}{2\pi}\int_{[\nu^2,\infty)}\frac{f(\lambda)\,d\lambda}{\lambda(\lambda-\nu^2)^{1/2}},$$

respectively, by $F'(\nu)=\frac{1}{\pi}\int_{[\nu^2,\infty)}\frac{f'(\lambda)\,d\lambda}{(\lambda-\nu^2)^{1/2}}$.

A Besov space consideration yields that

$\big[F(A_+)-F(A_-)\big]\in\mathcal{B}_1(\mathcal{H})$ if

$(1+\nu^2)^{3/4}F'\in L^2(\mathbb{R};d\nu),\ (1+\nu^2)^{9/4}\big|F''+3\nu(1+\nu^2)^{-1}F'\big|\in L^2(\mathbb{R};d\nu),$

and $F'\in L^\infty(\mathbb{R};d\nu)$. Here's the corresponding **heat kernel** version:

$$f(\lambda)=e^{-s\lambda},\quad F(\nu)=-\tfrac{1}{2}\operatorname{erf}\big(s^{1/2}\nu\big),\ \ s\in(0,\infty),$$

where $\operatorname{erf}(x)=\frac{2}{\pi^{1/2}}\int_0^x dy\,e^{-y^2}$, $x\in\mathbb{R}$.

Applications: Fredholm Indices, contd.

Note: $\mathbf{D_A} = (d/dt) + \mathbf{A}$ in $L^2(\mathbb{R};\mathcal{H})$ is a **model operator**: It arises in connection with **Dirac-type operators** (on compact and noncompact manifolds), the Maslov index, Morse theory (index), Floer homology, winding numbers, Sturm oscillation theory, dynamical systems, etc.

The literature on the **spectral flow and index theory** is endless:

M. F. Atiyah, N. Azamov, M.-T. Benameur, B. Booss-Bavnbek, N. V. Borisov, C. Callias, A. Carey, P. Dodds, K. Furutani, P. Kirk, M. Lesch, W. Müller, N. Nicolaescu, J. Phillips, V. Patodi, A. Pushnitski, P. Rabier, A. Rennie, J. Robin, D. Salamon, R. Schrader, I. Singer, F. Sukochev, C. Wahl, K. P. Wojciechowski, etc.

Just scratching the surface ....

Applications: Fredholm Indices, contd.

Just a few selections:

C. Callias, *Axial anomalies and index theorems on open spaces*, Commun. Math. Phys. **62**, 213–234 (1978). **Started an avalanche in Supersymmetric QM.**

D. Bolle, F.G., H. Grosse, W. Schweiger, and B. Simon, *Witten index, axial anomaly, and Krein's spectral shift function in supersymmetric quantum mechanics*, J. Math. Phys. **28**, 1512–1525 (1987).

This treats the scalar case when $A(t)$ is a scalar function and hence $\dim(\mathcal{H}) = 1$ (humble beginnings!). The **Krein–Lifshitz spectral shift function** is **linked to index theory**. See also,

R. W. Carey and J. D. Pincus, Proc. Symp. Pure Math. 44, 149–161 (1986),

W. Müller, Springer Lecture Notes in Math. Vol. **1244** (1987).

J. Robbin and D. Salamon, *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27**, 1–33 (1995). **A very influential paper.**

A. Pushnitski, *The spectral flow, the Fredholm index, and the spectral shift function*, in *Spectral Theory of Differential Operators: M. Sh. Birman 80th Birthday Collection*, AMS, 2008, pp. 141–155. **Motivated our recent work.**

Applications: The Witten Index

Non-Fredholm case: We will **not** assume $\boxed{0 \in \rho(A_+) \cap \rho(A_-)}$ and drop the requirement that $\mathbf{D_A}$ is Fredholm:

Definition.

Let T be a closed, linear, densely defined operator in $\mathcal{H}$ and suppose that for some $z \in \mathbb{C}\backslash[0,\infty)$,

$$(TT^* - z\, I_{\mathcal{H}})^{-1} - (T^*T - z\, I_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H}).$$

Introduce the **resolvent regularization**

$$\Delta(T,z) = z\, \mathrm{tr}_{\mathcal{H}}\left((TT^* - z\, I_{\mathcal{H}})^{-1} - (T^*T - z\, I_{\mathcal{H}})^{-1}\right), \quad z \in \mathbb{C}\backslash[0,\infty),$$

and sectors $S_\varphi := \{z \in \mathbb{C} \,|\, |\arg(z)| < \varphi\}$. Fix $\varphi \in (0,\pi/2)$, then the **Witten index** $W_r(T)$ of T is defined by

$$W_r(T) = \lim_{z\to 0,\, z\in\mathbb{C}\backslash S_\varphi} \Delta(T,z),$$

whenever this limit exists.

The subscript "r" indicates the use of the **resolvent regularization.**

Applications: The Witten Index

Analogously we could use a **heat kernel regularization**, replacing $\Delta(T, z)$ by $\left[e^{-tTT^*} - e^{-tT^*T}\right]$, $t > 0$, and study the limit $t \to \infty$.

For brevity, we'll **restrict** ourselves to the **resolvent regularization** in this talk.

Consistency with the **Fredholm index**:

Theorem.

Suppose that T is a Fredholm operator in $\mathcal{H}$. Then the Witten index $W_r(T)$ exists, equals the Fredholm index, $\operatorname{ind}(T)$, of T, and

$$W_r(T) = \operatorname{ind}(T) = \xi(0_+; TT^*, T^*T).$$

Note. In general (i.e., if T is not Fredholm), $W_r(T)$ is **not** integer-valued, it can be **any real number**. In concrete 2d magnetic field systems it can have the meaning of (non-quantized) magnetic flux $F \in \mathbb{R}$.

F.G. and B. Simon, *Topological invariance of the Witten index*, J. Funct. Anal. **79**, 91–102 (1988), showed that $W_r(T)$ has **stability properties** w.r.t. additive perturbations similar to the Fredholm index, replacing the **relative compactness** assumption on the perturbation by appropriate **relative trace class** conditions.

Applications: The Witten Index, contd., $\dim(\mathcal{H}) < \infty$

Now we return to the **model operator** $\mathbf{D_A} = (d/dt) + \mathbf{A}$ in $L^2(\mathbb{R}; \mathcal{H})$:

(1) **Special Case:** $\dim(\mathcal{H}) < \infty$.

- Assume A_- is a self-adjoint matrix in $\mathcal{H}$.

- Suppose there exist families of self-adjoint matrices $\{B(t)\}_{t\in\mathbb{R}}$ such that $B(\cdot)$ is locally absolutely continuous on $\mathbb{R}$.

- Assume that $\int_{\mathbb{R}} dt \, \|B'(t)\|_{\mathcal{B}(\mathcal{H})} < \infty$.

In the special case $\dim(\mathcal{H}) < \infty$ a complete picture emerges:

First, we note

Lemma.

Under the new set of hypotheses for $\dim(\mathcal{H}) < \infty$, $\mathbf{D_A}$ (equivalently, $\mathbf{D_A^*}$) is **Fredholm** if and only if $0 \notin \{\sigma(A_+) \cup \sigma(A_-)\}$.

Applications: The Witten Index, contd., $\dim(\mathcal{H}) < \infty$

The **non-Fredholm** case if $\dim(\mathcal{H}) < \infty$:

Theorem. (A. Carey, F.G., D. Potapov, F. Sukochev, Y. Tomilov, in prep.)

Assume the new set of hypotheses for $\dim(\mathcal{H}) < \infty$. Then $\xi(\,\cdot\,; \mathbf{H}_2, \mathbf{H}_1)$ has a continuous representative on the interval $(0, \infty)$, $\xi(\,\cdot\,; A_+, A_-)$ is piecewise constant a.e. on $\mathbb{R}$, the Witten index $W_r(\mathbf{D_A})$ exists, and

$$\begin{aligned} W_r(\mathbf{D_A}) &= \xi(0_+; \mathbf{H}_2, \mathbf{H}_1) \\ &= [\xi(0_+; A_+, A_-) + \xi(0_-; A_+, A_-)]/2 \\ &= \tfrac{1}{2}[\#_>(A_+) - \#_>(A_-)] - \tfrac{1}{2}[\#_<(A_+) - \#_<(A_-)]. \end{aligned}$$

In particular, in the finite-dimensional context, $\dim(\mathcal{H}) < \infty$, $W_r(\mathbf{D_A})$ is either an **integer**, or a **half-integer** (i.e., a **Levinson-type theorem** in scattering theory).

Here $\#_>(A)$ (resp. $\#_<(A)$) denotes the number of strictly positive (resp., strictly negative) eigenvalues of a self-adjoint operator A in $\mathcal{H}$, counting multiplicity.

Details rely on scattering theoretic techniques.

Applications: The Witten Index, contd., $\dim(\mathcal{H}) = \infty$

(2) **General Case:** $\dim(\mathcal{H}) = \infty$**.** Back to our **Main Hypotheses:**

- A_- – **self-adjoint** on $\mathrm{dom}(A_-) \subseteq \mathcal{H}$, $\mathcal{H}$ a complex, separable Hilbert space.
- $B(t)$, $t \in \mathbb{R}$, – **closed, symmetric,** in $\mathcal{H}$, $\mathrm{dom}(B(t)) \supseteq \mathrm{dom}(A_-)$.
- There exists a family $B'(t)$, $t \in \mathbb{R}$, – **closed, symmetric,** in $\mathcal{H}$, with

  $\mathrm{dom}(B'(t)) \supseteq \mathrm{dom}(A_-)$, such that

  $B(t)(|A_-| + I_{\mathcal{H}})^{-1}$, $t \in \mathbb{R}$, is **weakly locally a.c.** and for a.e. $t \in \mathbb{R}$,

  $$\frac{d}{dt}\big(g, B(t)(|A_-| + I_{\mathcal{H}})^{-1}h\big)_{\mathcal{H}} = \big(g, B'(t)(|A_-| + I_{\mathcal{H}})^{-1}h\big)_{\mathcal{H}}, \quad g, h \in \mathcal{H}.$$

- $B'(\cdot)(|A_-| + I_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H})$, $t \in \mathbb{R}$, $\int_{\mathbb{R}} \big\| B'(t)(|A_-| + I_{\mathcal{H}})^{-1} \big\|_{\mathcal{B}_1(\mathcal{H})}\, dt < \infty$.
- $\big\{(|B(t)|^2 + I_{\mathcal{H}})^{-1}\big\}_{t \in \mathbb{R}}$ and $\big\{(|B'(t)|^2 + I_{\mathcal{H}})^{-1}\big\}_{t \in \mathbb{R}}$ are **weakly measurable.**

Principal objects: $A(t) = A_- + B(t)$, $t \in \mathbb{R}$, and $\mathbf{A} = \int_{\mathbb{R}}^{\oplus} A(t)\, dt$ in $L^2(\mathbb{R}; \mathcal{H})$.

Applications: The Witten Index, contd., $\dim(\mathcal{H}) = \infty$

The **non-Fredholm** case if $\dim(\mathcal{H}) = \infty$:

A first fact:

For $\varphi \in (0, \pi/2)$ we introduce the sector

$$S_\varphi := \{z \in \mathbb{C} \,|\, |\arg(z)| < \varphi\}. \tag{3.1}$$

Theorem. (A. Carey, F.G., D. Potapov, F. Sukochev, Y. Tomilov, in prep.)

Assume the general hypotheses for $\dim(\mathcal{H}) = \infty$ and let $\varphi \in (0, \pi/2)$ be fixed. If 0 is a **right and left Lebesgue point** of $\xi(\,\cdot\,; A_+, A_-)$, then

$$\begin{aligned} W_r(\mathbf{D_A}) &= \lim_{z \to 0,\, z \in \mathbb{C}\backslash S_\varphi} z \, \mathrm{tr}_{L^2(\mathbb{R};\mathcal{H})} \big((\mathbf{H}_2 - z\,\mathbf{I})^{-1} - (\mathbf{H}_1 - z\,\mathbf{I})^{-1}\big) \\ &= - \lim_{z \to 0,\, z \in \mathbb{C}\backslash S_\varphi} z \int_{\mathbb{R}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\nu^2 - z)^{3/2}} \\ &= \frac{1}{2}[\xi(0_+; A_+, A_-) + \xi(0_-; A_+, A_-)]. \end{aligned}$$

Applications: The Witten Index, contd., $\dim(\mathcal{H}) = \infty$

Naturally, the proof relies on a series of careful estimates employing Pushnitski's formula,

$$\xi(\lambda; \mathbf{H}_2, \mathbf{H}_1) = \frac{1}{\pi} \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\lambda - \nu^2)^{1/2}} \text{ for a.e. } \lambda > 0,$$

and the trace identity,

$$\operatorname{tr}_{L^2(\mathbb{R};\mathcal{H})} \big((\mathbf{H}_2 - z\,\mathbf{I})^{-1} - (\mathbf{H}_1 - z\,\mathbf{I})^{-1} \big) = \frac{1}{2z} \operatorname{tr}_{\mathcal{H}} (g_z(A_+) - g_z(A_-)),$$

where $g_z(x) = \frac{x}{\sqrt{x^2 - z}}$, $\quad z \in \mathbb{C} \backslash [0, \infty)$, $x \in \mathbb{R}$, a smoothed-out sign fct.

Neither formula depends on the Fredholm property of $\mathbf{D_A}$.

Applications: The Witten Index, contd., $\dim(\mathcal{H}) = \infty$

Lebesgue points: Let $f \in L^1_{\text{loc}}(\mathbb{R}; dx)$.

Then $x_0 \in \mathbb{R}$ is a **right Lebesgue point of f** if there exists an $\alpha_+ \in \mathbb{C}$ such that

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{x_0}^{x_0+h} |f(x) - \alpha_+| \, dx = 0. \qquad \text{One then defines } f(x_{0,+}) = \alpha_+.$$

Similarly, $x_0 \in \mathbb{R}$ is a **left Lebesgue point of f** if there exists an $\alpha_- \in \mathbb{C}$ such that

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{x_0-h}^{x_0} |f(x) - \alpha_-| \, dx = 0. \qquad \text{One then defines } f(x_{0,-}) = \alpha_-.$$

Finally, $x_0 \in \mathbb{R}$ is a **Lebesgue point of f** if there exist $\alpha \in \mathbb{C}$ such that

$$\lim_{h \downarrow 0} \frac{1}{2h} \int_{x_0-h}^{x_0+h} |f(x) - \alpha| \, dx = 0. \qquad \text{One then defines } f(x_0) = \alpha.$$

That is, $x_0 \in \mathbb{R}$ is a **Lebesgue point of f** if and only if it is a **left and a right Lebesgue point** and $\alpha_+ = \alpha_- = \alpha$.

Applications: The Witten Index, contd., $\dim(\mathcal{H}) = \infty$

A second fact:

Theorem. (A. Carey, F.G., D. Potapov, F. Sukochev, Y. Tomilov, in prep.)

Assume the general hypotheses for $\dim(\mathcal{H}) = \infty$. If 0 is a **right and left Lebesgue point** of $\xi(\,\cdot\,; A_+, A_-)$, then it is a **right Lebesgue point** of $\xi(\,\cdot\,; \mathbf{H}_2, \mathbf{H}_1)$ and

$$\xi(0_+; \mathbf{H}_2, \mathbf{H}_1) = \frac{1}{2}[\xi(0_+; A_+, A_-) + \xi(0_-; A_+, A_-)].$$

The proof employs Pushnitski's formula,

$$\xi(\lambda; \mathbf{H}_2, \mathbf{H}_1) = \frac{1}{\pi} \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\lambda - \nu^2)^{1/2}} \text{ for a.e. } \lambda > 0,$$

and combines the **right/left Lebesgue point** property of $\xi(\,\cdot\,; A_+, A_-)$ at 0 with Fubini's theorem:

Applications: The Witten Index, contd., $\dim(\mathcal{H}) = \infty$

Sketch of Proof.

Since $\chi_{[-\sqrt{\lambda},\sqrt{\lambda}]}(\nu)\frac{1}{\lambda-\nu^2}$ is **even** w.r.t. $\nu \in \mathbb{R}$, and thus

$$\begin{aligned}
&\xi(\lambda; \mathbf{H}_2, \mathbf{H}_1) - \frac{1}{2}[\xi(0_+; A_+, A_-) + \xi(0_-; A_+, A_-)] \\
&= \frac{1}{\pi}\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{\xi(t; A_+, A_-)\, d\nu}{\sqrt{\lambda-\nu^2}} - \frac{1}{2}[\xi(0_+; A_+, A_-) + \xi(0_-; A_+, A_-)] \\
&= \frac{1}{\pi}\int_0^{\sqrt{\lambda}} \frac{[\xi(\nu; A_+, A_-) - \xi(0_-; A_+, A_-)] + [\xi(-\nu, A_+, A_-) - \xi(0_-; A_+, A_-)]}{\sqrt{\lambda-\nu^2}}\, d\nu.
\end{aligned}$$

Next, let 0 be a **right** and a **left Lebesgue point** of $\xi(\,\cdot\,; A_+, A_-)$, then abbreviating

$$f_\pm(\nu) := \xi(\pm\nu; A_+, A_-) - \xi(0_\pm; A_+, A_-), \quad \nu \in \mathbb{R},$$

and applying Fubini's theorem yields,

Applications: The Witten Index, contd., $\dim(\mathcal{H}) = \infty$

$$
\begin{aligned}
&\lim_{h\downarrow 0+} \frac{1}{h}\int_0^h \bigg|\xi(\lambda;\mathbf{H}_2,\mathbf{H}_1) - \frac{1}{2}[\xi(0_+;A_+,A_-) + \xi(0_-;A_+,A_-)]\bigg|\,d\lambda \\
&\quad = \lim_{h\downarrow 0+} \frac{1}{\pi h}\int_0^h \bigg|\int_0^{\sqrt{\lambda}} \frac{[f_+(\nu) + f_-(\nu)]\,d\nu}{\sqrt{\lambda-\nu^2}}\bigg|\,d\lambda \\
&\quad \leq \lim_{h\downarrow 0+} \frac{1}{\pi h}\int_0^h \bigg(\int_0^{\sqrt{\lambda}} \frac{[|f_+(\nu)| + |f_-(\nu)|]\,d\nu}{\sqrt{\lambda-\nu^2}}\bigg)\,d\lambda \\
&\quad = \lim_{h\downarrow 0+} \frac{1}{\pi h}\int_0^{\sqrt{h}} [|f_+(\nu)| + |f_-(\nu)|]\bigg(\int_{\nu^2}^h \frac{d\lambda}{\sqrt{\lambda-\nu^2}}\bigg)\,d\nu \\
&\quad = \lim_{h\downarrow 0+} \frac{2}{\pi h}\int_0^{\sqrt{h}} [|f_+(\nu)| + |f_-(\nu)|]\sqrt{h-\nu^2}\,d\nu \\
&\quad = \lim_{h\downarrow 0+} \frac{2}{\pi \sqrt{h}}\int_0^{\sqrt{h}} [|f_+(\nu)| + |f_-(\nu)|]\sqrt{1-[\nu^2/h]}\,d\nu \\
&\quad \leq \lim_{h\downarrow 0+} \frac{2}{\pi \sqrt{h}}\int_0^{\sqrt{h}} [|f_+(\nu)| + |f_-(\nu)|]\,d\nu = 0 \quad \text{by the } \textbf{right/left L-point} \text{ hyp.}
\end{aligned}
$$

Applications: The Witten Index, contd., $\dim(\mathcal{H}) = \infty$

Combining these results yields:

Theorem. (A. Carey, F.G., D. Potapov, F. Sukochev, Y. Tomilov, in prep.)

Assume the general hypotheses for $\dim(\mathcal{H}) = \infty$ and that 0 is a **right and left Lebesgue point** of $\xi(\,\cdot\,; A_+, A_-)$ (and hence a **right Lebesgue point** of $\xi(\,\cdot\,; \mathbf{H}_2, \mathbf{H}_1)$). Then, for fixed $\varphi \in (0, \pi/2)$,

$$\begin{aligned} W_r(\mathbf{D_A}) &= \lim_{z \to 0,\, z \in \mathbb{C} \setminus S_\varphi} z \operatorname{tr}_{L^2(\mathbb{R};\mathcal{H})} \big((\mathbf{H}_2 - z\,\mathbf{I})^{-1} - (\mathbf{H}_1 - z\,\mathbf{I})^{-1} \big) \\ &= \xi(0_+; \mathbf{H}_2, \mathbf{H}_1) \\ &= - \lim_{z \to 0,\, z \in \mathbb{C} \setminus S_\varphi} z \int_{\mathbb{R}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\nu^2 - z)^{3/2}} \\ &= \frac{1}{2} [\xi(0_+; A_+, A_-) + \xi(0_-; A_+, A_-)]. \end{aligned}$$

Final Thoughts:

A VERY HAPPY (belated) BIRTHDAY, YOSI!