COUNTING LATTICE POINTS IN THE PLANE

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GOALS

The goals of this short paper is to discuss Hlawaka’s bound for counting lattice points in the plane.

1. GAUSS’ THEOREM

Theorem. (Gauss) - Define \( B_n(R) = \{ v \in \mathbb{Z}^n \mid \|v\| \leq R \} \), then \( B_n(R) \sim R^n \), more accurately -

\[
B_n(R) = \frac{\pi^{n/2}}{\Gamma(1+n/2)} R^n + O(R^{n-1}).
\]

The proof is trivial.

By central limit considerations, we expect much better cancellations in the reminder term.

We will present a short modification of this counting in two-dimensions, due to Hlawaka, which gives a better bound for the error - \( O(R^{2/3}) \).

2. FOURIER TRANSFORM - STATIONARY PHASE

Here we give some estimates for the actual decay of some Fourier transforms.

Define \( \chi(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \) and \( \chi_R(x) := \chi(x/R) \).

Lemma 1. \( \hat{\chi}_R(z) = R^2 \cdot \hat{\chi}(Rz) \).

The proof is a trivial computation.

We will be interested in the asymptotics of \( \hat{\chi}(Rz) \).

Lemma 2. Fourier transform of a spherical function is spherical.
Proof. Let $f(x) = f(|x|)$, then $\hat{f}(z) = \int_{x \in \mathbb{R}^n} f(x) e^{2\pi i (z,x)} dx$, by changing to polar coordinates - $\hat{f}(z) = \int_{\rho=0}^{\infty} \int_{v \in S^{n-1}} f (\rho \cdot v) e^{2\pi i \rho (v,z)} f(v) dv d\rho = \int_{\rho=0}^{\infty} f (\rho) \rho^{n-1} d\rho \int_{v \in S^{n-1}} e^{2\pi i \rho (v,z)} j(v) dv$, where $j(v) = J(v, \rho)/\rho^{n-1}$.

Definition 3. The $v$’th Bessel function is defined as - $J_v(t) = \frac{(t/2)^v}{\Gamma(v+1/2)\Gamma(1/2)} \int_{-1}^{1} e^{i\pi v} (1 - s^2)^{-1/2} ds$.

Theorem 4. $\hat{\chi}(z) \sim |z|^{-1} J_1 (2\pi |z|)$.

Proof. As the function is spherical, we will compute the Fourier transform at the $x$ -axis, namely $z = (z, 0)$.

$$\int_{v \in \mathbb{R}^2} e^{2\pi i (v,z)} dm(v) = \int_{v_1 = -1}^{1} \int_{v_2 = -\sqrt{1-v_1^2}}^{\sqrt{1-v_1^2}} e^{2\pi i z v_1} dv_2 dv_1 = 2 \int_{-1}^{1} \sqrt{1-v_1^2} e^{2\pi i z v_1} dv_1 = C |z|^{-1} J_1 (2\pi z).$$

Remark. Notice that $1 = \frac{n}{2}$, in general we expect the decay rate of a Fourier coefficient of “nice” $k$ -dimensional body in $\mathbb{R}^n$ to be of order $O \left( |z|^{-\frac{n}{2}} \right)$, by central limit.

Example 5. We would like to discover a typical example of the stationary phase method. Fix $m, n$ not both 0, and look at the following integral (as parameter of $r$) - $\int_{0}^{1} e^{2\pi i (mr^2 + nx)} dx$. We would like to show that $\int_{0}^{1} e^{2\pi i (mr^2 + nx)} dx \to 0$ as $r \to \infty$, in a quantitative manner.

$$\int_{0}^{1} e^{2\pi i (mr^2 + nx)} dx \sim \int_{0}^{1} e^{2\pi i (mr^2 + nx)} \frac{(2mr+n)}{2mr+nr} dx.$$  

If $x \neq -\frac{n}{2m}$, we could have used integration by parts in the following manner - $\int_{0}^{1} e^{2\pi i (mr^2 + nx)} \frac{(2mr+n)}{(2mr+nr)} dx = \left[ e^{2\pi i (mr^2 + nx)} \right]_{0}^{1} + \int_{0}^{1} 2mr e^{2\pi i (mr^2 + nx)} \frac{2mr+n}{2mr+nr} dx$. The first term is bounded by $\frac{2}{\min \{2mr+n \}_{x \in [0,1]}}$, and the second term is bounded by $2 |m| \frac{1}{\min \{2mr+n \}_{x \in [0,1]}}$, and we get $O \left( \frac{4m}{r} \right)$ estimate for the integral.

But it may well happen that $x = -\frac{n}{2m}$, and we can’t write the integral in the preceding form.

So fix $\varepsilon$-neighborhood around $x = -\frac{n}{2m}$ and dissect the integral into 2 parts - $\int_{\frac{n}{2m}}^{\frac{n}{2m}+\varepsilon} + \int_{|x|+\frac{m}{2m}}^{\varepsilon}$, the first integral is clearly bounded by $2\varepsilon$, where for the second term, we may use the preceding
computation, but we get \( \min \{ |2mx + n| \}_{m \neq \frac{n}{m}} > \) to be at-least \( 2m \varepsilon \), and we get in the estimate \( O \left( \frac{2}{m^2} \right) \).

Now picking \( \varepsilon \) to be a small power of \( r \), \( \varepsilon = r^{-\alpha} \) we get \( r^{-\alpha} + r^{\alpha-1} \) estimate, choosing \( \alpha = 1/2 \) we get the optimized rate.

**Corollary 6.** Define \( \gamma(t) : [0,1] \rightarrow \mathbb{R}/\mathbb{Z} \) by \( \gamma(t) = (t,t^2) \). The family of curves \( \{ r \cdot \gamma(t) \}_{r \in \mathbb{R}^+} \) gets equidistributed (wrt the Lebesgue measure) as \( r \rightarrow \infty \).

**Proof.** Apply Weyl’s criterion with the preceding example. \( \qed \)

**Remark 7.** For general \( f \), when considering integral as \( \int e^{irf(x)} dx \), expand \( f(x) \) by Taylor series near the stationary points, and use the estimate which can be derived for \( e^{ikx^2} \). For multivariable version, expand the Taylor approximation to quadratic form using the Hessian and conjugate the Hessian to deal with a diagonal quadratic, hence the integral turns to a product over the different dimensions of the one-dimensional case.

**Theorem 8.** *(Stationary Phase estimate)* - \( J_1(z) \sim |z|^{-1/2} \) for \( z \gg 0 \).

**Proof.** We will use the following presentation of the Bessel functions - \( J_v(t) = \int_0^1 \cos (v \pi \theta - t \sin \pi \theta) d \theta = \Re \int_0^1 e^{v \pi i \theta - it \sin (\pi \theta)} d \theta \).

We can allegedly rewrite the integral as follows -

\[
\int_0^1 e^{v \pi i \theta - it \sin (\pi \theta)} d \theta = \int_0^1 e^{v \pi i \theta - it \sin (\pi \theta)} \frac{v \pi i - it \cos (\pi \theta)}{v \pi i - it \cos (\pi \theta)} d \theta, 
\]

if \( v \pi i - it \cos (\pi \theta) \neq 0 \) (or equivalently, \( \cos (\pi \theta) \neq \frac{v}{t} \)).

If so, we could have used integration by parts -

\[
e^{v \pi i \theta - it \sin (\pi \theta)} \frac{v \pi i - it \cos (\pi \theta)}{v \pi i - it \cos (\pi \theta)} = e^{v \pi i \theta - it \sin (\pi \theta)} \frac{it \sin (\pi \theta)}{(v \pi i - it \cos (\pi \theta))^2} d \theta 
\]

, where we get a bound of \( O(1/t) \).

The trouble is that we have points for which \( \cos (\pi \theta) = \frac{v}{t} \), exactly two of those for \( t \gg v \).
So circle those points \( \theta_0 = \frac{1}{2} \arccos \left( \frac{1}{r} \right), \ 1 - \frac{1}{2} \arccos \left( \frac{1}{r} \right) \) with \( \varepsilon \)-neighborhoods. outside of those neighborhoods we have \( |v \pi i - it \cos (\pi \theta)| > \delta \) and therefore we get an estimate of the form \( O(\varepsilon) + O(1/\delta t) \).

Now we investigate the relationship between \( \delta \) and \( \varepsilon \) more closely, define \( f(\theta) = \cos (\pi \theta) - \frac{1}{r}, \ \ f'(\theta) = -\pi \sin (\pi \theta), \) recall by Pythagorean theorem we have \( \sin (\arccos (\beta)) = \sqrt{1 - \beta^2} \) therefore \( f'(\theta_0) = \sqrt{1 - \frac{c^2}{\pi^2}} \), therefore we approximate \( f(\theta) \) by \( f(\theta) \sim 1 \cdot (\theta - \theta_0) \) as \( t \to \infty \) (notice we can easily get \( |f(\theta)| > C|\theta - \theta_0| \) for some explicit \( C \) as long \( t > t_0 \)).

Therefore, at \( \varepsilon \)-neighborhoods, we get \( C \varepsilon \) difference, hence \( \left| (v \pi i \theta - it \sin (\pi \theta))' \right| > C \varepsilon \) for \( \theta \) outside of those \( \varepsilon \)-neighborhoods.

Hence we get \( \delta = C \varepsilon \).

Pick \( \varepsilon = t^{-\alpha} \) and get \( t^{-\alpha} = t^{\alpha - 1} \) therefore we choose \( \alpha = 1/2 \) and deduce the required bound.

\( \square \)

**Corollary 9.** \( \hat{\chi}(z) \sim |z|^{-3/2} \) as \( z \to \infty \).

### 3. Fourier Transform - Poisson Summation

Assume that \( f \) is a Schwartz function on \( \mathbb{R} \) for now.

As a result of rapid decay, we can define \( \tilde{f}(x) = \sum_{n \in \mathbb{Z}} f(x + n) \), which converges everywhere and uniformly, and the function is bounded by say \( \sum_{n \in \mathbb{Z}} \max |f||_{[n,n+1]} \), which is less than \( K \sum_{n \in \mathbb{Z}} \frac{1}{n^2} \) for appropriate \( K \), because \( f \) is rapidly decreasing.

**Exercise.** \( \tilde{f} \) is invariant under integer translation (\( \tilde{f} \) is 1-periodic function).

Therefore, we can treat \( \tilde{f} \) as a nice function of \( \mathbb{R}/\mathbb{Z} \), in particular \( \tilde{f} \in L^2(\mathbb{R}/\mathbb{Z}) \), as a result, we can expend \( \tilde{f} \) to a Fourier series, \( \tilde{f} = \sum_{n \in \mathbb{Z}} c_n e_n(x) \). Due to the fact that \( \tilde{f} \) is also differentiable, we have by Dirichlet’s theorem that the Fourier series converge to \( \tilde{f} \) in the pointwise manner.

In particular, if we evaluate the equality at \( x = 0 \) we get \( \tilde{f}(0) = \sum_{n \in \mathbb{Z}} c_n \), where \( \tilde{f}(0) = \sum_{n \in \mathbb{Z}} f(n) \).

On the other hand, by unfolding we get - \( c_k = \int_0^1 \tilde{f}(x) e^{2\pi ikx} dx = \int_0^1 \sum_{n \in \mathbb{Z}} f(x+n) e^{2\pi ikx} dx = \int_{-\infty}^{\infty} f(x) e^{2\pi ikx} dx = \hat{f}(k) \), therefore \( \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) \).

The same principle is true in \( \mathbb{R}^n \) and we get \( \sum_{\mathbb{N} \in \mathbb{Z}^n} f(\vec{n}) = \sum_{\mathbb{N} \in \mathbb{Z}^n} \hat{f}(\vec{n}) \).
4. Principles of point counting by Fourier Analysis

We are utilizing the two-dimensional analogue of Poisson summation.

Notice that \( B_2(R) = \sum_{n \in \mathbb{Z}^2} \chi_R(n) \). Notice that \( \chi_R \) is not a “valid” function for Poisson summation according to our proof, this can be fixed if we assume some decay condition at infinity for \( \max \{|f|, |\hat{f}|\} \) showing the resulting Fourier series involved in the proof of the summation formula indeed converge absolutely and uniformly. But we will mollify the function shortly, having a smoothed approximation which allows us to use Poisson summation.

Therefore we can infer from the Poisson summation formula -

\[
\sum_{n \in \mathbb{Z}^2} \chi_R(n) = \sum_{n \in \mathbb{Z}^2} \hat{\chi}_R(n) = \sum_{n \in \mathbb{Z}^2} R^2 \hat{\chi}(Rn) = R^2 \langle \chi \rangle + R^2 \sum_{n \in \mathbb{Z}^2} \hat{\chi}(Rn)
\]

Notice that the second term does not converge (one can take \( n = (n, 0) \) and get divergence).

5. Uncertainty and Mollification

Let \( \phi(x) \) be a nice smooth bump function of width \( H \) to be chosen later, with integral one, and we define the modified counting function \( B_2^H(R) = \sum_{n \in \mathbb{Z}^2} \chi_R * \phi(n) \).

Assume that \( \phi_1(x) \) is with width 1, \( \phi_1(z) \sim |z|^{-N} \), then \( \hat{\phi}(z) \sim H^{-2}H^2 |Hz|^{-N} = H^{-N} |z|^{-N} \)

As such smoothing, we are adding some additional counting in a \( H \)-neighbourhood of the boundary of \( B_R \), in particular if the main term we expect is \( \pi R^2 \), for \( B_2^H(R) \) we expect something between \( \pi R^2 \) and \( \pi (R+H)^2 \).

On the other hand, by the convolution identity, we have \( \hat{\chi_R * \phi} = \hat{\chi_R} \cdot \hat{\phi} \).

Notice that in the large modes,
\[
\sum_{|\pi| > H^{-1}}' \hat{\chi}_{\rho}(z) \cdot \hat{\phi}(\pi) = R^2 \sum_{|\pi| > H^{-1}}' \hat{\chi}(R \pi) \cdot \hat{\phi}(\pi) \sim R^2 \sum_{|\pi| > H^{-1}} \frac{1}{R^{3/2} |\pi|^{3/2}} H^{-N} |\pi|^{-N}
\]
\[
= R^{1/2} H^{-N} \sum_{|\pi| > H^{-1}} |\pi|^{-3/2-N} \sim R^{1/2} H^{-N} \int_{\rho = H^{-1}}^\infty \rho \cdot \rho^{-3/2-N} d\rho \quad (N \geq 1, \text{ for integrability})
\]
\[
= R^{1/2} H^{-N} \int_{\rho = H^{-1}}^\infty \rho^{-1/2-N} d\rho = R^{1/2} H^{-N} \left( H^{-1} \right)^{1/2-N} = R^{1/2} H^{-1/2}
\]

For the small modes,
\[
\sum_{|\pi| < H^{-1}}' \hat{\chi}_{\rho}(\pi) \cdot \hat{\phi}(\pi) \leq CR^2 \sum_{0 < |\pi| < H^{-1}} |\hat{\chi}(R \pi)| = CR^2 \sum_{0 < |\pi| < H^{-1}} R^{-3/2} |\pi|^{-3/2}
\]
\[
= CR^{1/2} \sum_{0 < |\pi| < H^{-1}} |\pi|^{-3/2} \sim CR^{1/2} \int_{\rho = 1}^{H^{-1}} \rho \cdot \rho^{-3/2} d\rho
\]
\[
\leq CR^{1/2} H^{-1/2}.
\]

Therefore we get \( B_2^H(R) \sim \pi R^2 + O \left( R^{1/2} H^{-1/2} \right) \).

Notice that the smoothing takes effect in neighborhood of size \( H \) of the boundary, so we have a boundary effect - \( |B_2^H(R) - B_2(R)| \leq O \left( 2\pi R \cdot H \right) \), hence we deduce \( B_2(R) = \pi R^2 + O \left( R^{1/2} H^{-1/2} \right) + O(RH) \), pick \( H = R^{-\alpha} \) and get \( R^{1/2 + \alpha/2} = R^{1 - \alpha} \).

So we choose \( 3\alpha/2 = 1/2 \) therefore \( \alpha = 1/3 \) and we have \( B_2(R) \sim \pi R^2 + O \left( R^{2/3} \right) \).

REFERENCES


