RELATIVE STANLEY–REISNER THEORY AND LOWER BOUND THEOREMS FOR MINKOWSKI SUMS

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Abstract. This note complements an earlier paper of the author by providing a lower bound theorem for Minkowski sums of polytopes.

In [AS16], we showed an analogue of McMullen’s Upper Bound theorem for Minkowski sums of polytopes, estimating the maximal complexity of such a sum of polytopes. A common question in reaction to that research was the question for an analogue for the Barnette’s Lower Bound Theorem in the same setting, and the purpose of this note is to put the question to rest.

Lower Bound Theorem for polytopes. For a simplicial $d$-dimensional polytope $P$ on $n$ vertices and $0 ≤ k < d$

$$f_k(P) ≥ f_k(\text{Stack}_d(n))$$

where $\text{Stack}_d(n)$ is a $d$-dimensional cyclic polytope on $n$ vertices. Moreover, equality holds for all $k$ whenever it holds for some $k_0$, $k_0 + 1 ≥ \lfloor \frac{d}{2} \rfloor$.

This theorem was proven by Barnette [Bar71], and is substantially deeper than the Upper Bound Theorem as it relates to the standard conjectures for toric varieties, see also [Kal87, MN13, Adi17].

In this paper we will address more general lower bound problems for polytopes and polytopal complexes. Recall that the Minkowski sum of polytopes $P, Q ⊆ \mathbb{R}^d$ is the polytope $P + Q = \{p + q : p ∈ P, q ∈ Q\}$, and bounding the complexity of Minkowski sums of polytopes is an important problem in several fields of mathematics, see [AS16] for a more thorough discussion of the applications. We thus address the question:

For given $k < d$ and $n_1, n_2, \ldots, n_m$, what is the minimal number of $k$-dimensional faces of the Minkowski sum $P_1 + P_2 + \cdots + P_m$ for polytopes $P_1, \ldots, P_m ⊆ \mathbb{R}^d$ with vertex numbers $f_0(P_i) = n_i$ for $i = 1, \ldots, m$?

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To make this problem nontrivial, it is useful to restrict to "general position" Minkowski sums of 'general position' polytopes, i.e., restrict to situations in which the Cayley complex is simplicial.

We adopt the notation of [AS16] for convenience.

1. Lower Bound Theorems for Minkowski Sums

Recall that for a family $P_{[m]}$ for $m$ simplicial $d$-polytopes in $\mathbb{R}^d$ in general position (also called a "pure collection"), the Cayley polytope is defined as

$$\text{Cay}(P_{[m]}) := \text{conv} \left( \bigcup_{i=1}^{m} P_i \times e_i \right) \subseteq \mathbb{R}^d \times \mathbb{R}^m.$$ 

The Cayley complex $T_{[m]} = T(P_{[m]})$ is obtained the Cayley polytope $\text{Cay}(P_{[m]})$ by removing the faces corresponding to $\text{Cay}(P_S)$ for $S \subseteq [m]$. Related, but distinct is the relative Cayley complex, which is the relative simplicial complex obtained as the pair

$$(\text{Cay}(P_{[m]}), \bigcup_{S \subseteq [m]} \text{Cay}(P_S)) =: T^o_{[m]} = T^o(P_{[m]})$$

We recall that, for relative complexes $\Psi$, we define

$$h_k(\Psi) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}(\Psi)$$

in terms of the number of $i$-dimensional faces $f_i$ and that

$$f_{i+m-1}(T^o_{[m]}) = f_i(|P_{[m]}|)$$

for $|P_{[m]}|$ the Minkowski sum of the polytopes in the family $P_{[m]}$.

It therefore remains to provide lower bounds on $h_{k+m-1}(T^o_{[m]})$, since Möbius inversion represents face numbers as a positive combination of $h$-numbers

$$f_{i-1}(\Psi) = \sum_{k=0}^{i} \binom{d-k}{i-k} h_k(\Psi).$$

**Lemma 1.1.** Let $P_{[m]}$ be a pure collection of $m$ polytopes in $\mathbb{R}^d$, $d \geq 1$, such that the Cayley complex $T(P_1, \ldots, P_m)$ is simplicial. Then, for all $-m + 1 \leq k \leq d$, we have

$$h_{d-k}(T_{[m]}) = \tilde{h}_{k+m-1}(T^o_{[m]}) := h_{k+m-1}(T^o_{[m]}) + (-1)^{k} \binom{d+m-1}{d-k}.$$ 

This is a reflection of Poincaré duality in $IH^*(\text{Cay}(P_{[m]}))$, where we leave out the fan over $\text{Cay}(P_{[m]})$ as an intermediate step in the construction.

This reduces the problem of bounding $h_{k+m-1}(T^o_{[m]})$ from below to the problem of bounding $h_{d-k}(T_{[m]})$ from below. We need the following consequence of the Hard Lefschetz theorem
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for $IH^*(\text{Cay}(P_m))$ (see [Kar04] for a combinatorial proof). Let $A^*(T_m)$ denote the Stanley–Reisner ring modulo Artinian reduction given by the coordinates of $\text{Cay}(P_m)$ in $\mathbb{R}^{d+m}$, or equivalently the Chow ring of the toric variety over the fan of $T_m$. Let $\omega$ denote the anticanonical divisor. The following proposition is crucial.

**Proposition 1.2.** Let $i \leq \frac{d}{2}$ and $j \leq d - 2i$. Then

$$A^i(T_m) \xrightarrow{\times \omega^j} A^{i+j}(T_m)$$

**Proof.** Let us first notice that

$$A^i(T_m) \cong IH^i(\text{Cay}(P_m))$$

where we use the polynomial grading for $IH$ inherited as a module of polynomial functions, see again [Kar04].

Note furthermore that $\omega^j$ vanishes on

$$\bigcup_{S \subseteq [m], \#S = j - 1} \text{Cay}(P_S)$$

so that we obtain an injection

$$A^i(T_m) \xrightarrow{\times \omega^j} A^{i+j}(T_m, \bigcup_{S \subseteq [m], \#S = j - 1} \text{Cay}(P_S))$$

Now, we briefly note that we may assume for simplicity that $j = d - 2i$, so that it remains to study the kernel of the map

$$(*) \quad A^{d-i}(T_m, \bigcup_{S \subseteq [m], \#S = j - 1} \text{Cay}(P_S)) \rightarrow A^{d-i}(T_m)$$

induced by inclusion. Define

$$T_j := \bigoplus_{S \subseteq [m], \#S = j - 1} (T_S),$$

which is naturally realized in $\mathbb{R}^{d+j-2}$, which we use to define the Artinian reduction of its Stanley–Reisner ring. Define also $B(T_j)$ as the $A(T_j)$ quotient obtained as

$$B^k(T_j) := \text{im} \left[ A^k(T_j) \rightarrow \bigoplus_{\sigma(d+j-2-k) - \text{face} \in T_j} A^k(\text{st}(\sigma, T_j)) \right]$$

then $\omega$ acts as a Lefschetz element, i.e. we have an isomorphism

$$B^k(T_j) \xrightarrow{\times \omega^{d+j-2-k}} B^{d+j-2-k}(T_j)$$

for $1 \leq k \leq \frac{d-j-2}{2}$, see [Kar04, Section 4.1]. The kernel of the map $(*)$ is then isomorphic to the map

$$A^{d-2i-1}(T_j) \xrightarrow{\times \omega} A^{d-2i}(T_j)$$
thus proving the claim.

As a consequence, we obtain:

**Corollary 1.3.** Let $i \leq \frac{d}{2}$ and $j \leq d - 2i$. Then

$$\dim A^i(T_{[m]}) \xrightarrow{\omega^j} \dim A^{i+j}(T_{[m]})$$

This, together with the work of Schenzel [Sch81] provides tight (generalized) lower bound theorems for Minkowski sums. Moreover, the tools of [?, AS16] characterize the case of equality.

**References**


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