TEST FUNCTIONS, PARTITIONING AND CONCENTRATION PHENOMENA IN TORIC GEOMETRY

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ABSTRACT. We discuss simple partitioning phenomena for intersection rings of certain toric varieties, and apply this to prove the following results.

- ◦ We generalize a result of Gräbe, proving that Stanley–Reisner rings of homology spheres are Gorenstein.
- ◦ We give a new and direct proof of a theorem of the authors due to which the cohomology rings of matroids are Poincare duality algebras.
- ◦ We prove a conjecture of Zharkov relating partitioning to Orlik–Solomon algebras of fans and matroids.
- ◦ We characterize simplicial polytopes with extremal primitive Betti numbers, and apply this to resolve a conjecture of Kalai concerning polytopes approximating smooth convex bodies.

1. INTRODUCTION

It is hardly necessary to motivate and convince anyone of the importance of Poincaré duality and the Gorenstein property. When investigating cohomology rings of matroids introduced by Feichtner–Yuzvinsky [FY04], we noticed that it also appears in recent proof of the Hodge–Riemann relations for general matroids in [AHK15]. There, we proved Poincaré duality using an inductive proof that was appropriate to the situation but rather non-enlightening as a general way to understand Poincaré duality in positive codimension.

We noticed, however, that it is closely related to certain partitioning properties for algebraic cycles in Chow rings, asserting that a Chow cohomology class can always be represented as a sum of cycles with small support, and completed also a more direct proof of Poincaré duality. At the same time, Ilya Zharkov asked us a very much related problem arising in the study for the Steenbrink spectral sequence for Hodge theory in smooth tropical varieties [IKMZ] that had remained open (though it could be circumvented for their purposes).

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Finally, we noticed that "partition of unity for algebraic cycles" is naturally related to toric chordality, and concentration properties for primitive cohomology classes in projective complete toric varieties, as investigated in [Adi15]. In particular, it could naturally be applied to open problems for simplicial polytopes.

2. STANLEY–REISNER THEORY

2.1. Let $\mathbb{K}$ be a field of characteristic 0 Δ is a simplicial complex on groundset $[n] := [1, \cdots, n]$, let $I_\Delta := \langle \mathbf{x}^{\mathbf{a}} : \operatorname{supp}(\mathbf{a}) \in \Delta \rangle$ denote the nonface ideal in $\mathbb{K}[\mathbf{x}]$, where $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \cdots, x_n]$. Let $A^*(\Delta) := \mathbb{K}[\mathbf{x}]/I_\Delta$ denote the Stanley–Reisner ring of Δ. A collection of linear forms $\Theta = (\theta_1, \cdots, \theta_\ell)$ in the polynomial ring $\mathbb{K}[\mathbf{x}]$ is a **linear system of parameters** if

$$\dim A^*(\Delta; \Theta) = \dim A^*(\Delta) - \ell$$

for $A^*(\Delta; \Theta) := A^*(\Delta)/\Theta A^*(\Delta)$ and $\dim$ the Krull dimension. If

$$\ell = \dim A^*(\Delta) = \dim \Delta + 1,$$

then Θ is simply **linear system of parameters**.

2.2. We can refine this definitions in several ways:

A **relative simplicial complex** is a pair of simplicial complexes $\Psi = (\Delta, \Gamma)$, $\Gamma \subset \Delta$. If $\Psi = (\Delta, \Gamma)$ is a relative simplicial complex, it is not hard to modify this definition to obtain the correct picture: we simply define $A^*(\Psi) = I_\Delta / I_\Gamma$. An important instance we will encounter again is the relative star $\mathrm{st}^\circ_F := (\mathrm{st}_F \Delta, \partial \mathrm{st}_F \Delta)$ of a face in a simplicial complex. Recall that the star $\mathrm{st}_F \Delta$ is the downward closure of the link of F in Δ, which is itself the filter generated by F in the poset of Δ.

Let $\Delta^{(k)}$ denote the collection of k-dimensional faces of a simplicial complex Δ Observe furthermore that Θ induces a map $\Delta^{(0)} \to \mathbb{K}^\ell$ by associating to the vertices of Δ the coordinates $V_\Delta = (v_1, \cdots, v_\ell)$, where $V_\Delta \mathbf{x} = \Theta \in \mathbb{K}^{\ell \times n}$. The ideal $\langle \Theta \rangle$ is therefore also called the **coordinate ideal**, which can for our purposes be identified with the family of linear forms Θ.

The reduced Stanley–Reisner rings, respectively, of a geometric simplicial complex (a simplicial complex with a map of the vertices into a vectorspace $\mathbb{K}^\ell$ are those given by the linear system of parameters given by the geometric realization. We simply will denote the coordinate ideal of a geometric complex by Θ_Δ.

2.3. A geometric simplicial complex in $\mathbb{K}^d$ is n-**proper** if the image of every k-face, ($k < n$), linearly spans a subspace of dimension $k+1$, and simply **proper** if $n = d$. A system of linear forms is a (partial) linear system of parameters if and only if the associated coordinatization is proper.

For a geometric simplicial complex, we shall generally leave out the system of linear forms in the definition of stress space and reduced Stanley–Reisner ring.

Finally, we say Θ is **regular** if, for every truncation $\Theta_{j-1} = (\theta_1, \cdots, \theta_{j-1})$, $j \leqslant \ell$, we have an injection

$$A(\Delta; \Theta_{j-1}) \xrightarrow{\times \theta_j} A(\Delta; \Theta_{j-1}).$$

for every i. The **depth** of a simplicial complex is defined as the length of the longest regular sequence that its stress space/Stanley–Reisner module admits. **Cohen–Macaulay** singles out the complexes/modules for which depth equals Krull dimension. We call a (geometric) simplicial complex combinatorially Cohen–Macaulay if its underlying abstract simplicial complex is Cohen–Macaulay.

2.4. The socle $\mathrm{Soc}(R)$ of a graded commutative algebra R over $\mathbb{K}$ is the collection of elements x in R such that $x \cdot R_1 = 0$, intuitively encoding the "maxima" under the poset order in a graded ring. If R is of Krull dimension zero, then is a **Gorenstein ring** if its socle is one-dimensional. If $\dim R > 0$, then R is called Gorenstein (over some field $\mathbb{K}$) if it is Cohen–Macaulay and its reduction by a linear system of parameters is a Gorenstein ring. Note that Gorenstein rings (of Krull dimension 0) are but the embodiment of Poincaré duality in commutative algebra: A graded Artinian $\mathbb{K}$-algebra R generated in degree one is a Poincare duality algebra if and only if R is Gorenstein (of dimension 0).

Naturally between Cohen–Macaulay and Gorenstein rings, we have **level** rings, introduced by Stanley [Sta77]. Similarly defined, a ring R of dimension 0 is level if its socle is concentrated in a single degree, and if it is Cohen–Macaulay and its reduction is a level ring otherwise. In other words, a level ring has a "pseudo-perfect pairing", albeit not with a unique fundamental class in the top degree.

3. Partition of unity

It is useful to note that, given us working in a field of characteristic 0, we can work with a rather specific model for the dual Stanley–Reisner ring $A_* := \mathrm{Hom}(A^*, \mathbb{K})$:

Consider $\mathbb{K}[\mathbf{x}]$ equipped with the standard inner product on monomials

$$\left\langle \mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \right\rangle := \frac{\delta_{\mathbf{a},\mathbf{b}}}{\mathbf{a}!\mathbf{b}!}$$

and define

$$S(\Delta)\ :=\ \langle I_\Delta\rangle^{\perp}\subset\mathbb{K}[\mathbf{x}]$$

Now, construct an action of the Stanley–Reisner ring on the stress space by observing that the adjoint operator to multiplication with a linear form $\theta=\theta(\mathbf{x})$ is the differential $\theta^{\vee}=\theta(\mathrm{d}/\mathrm{d}\mathbf{x})$; with this we have

$$S(\Delta;\Theta)\ :=\ \ker\left[\Theta^{\vee}:S(\Delta)\longrightarrow S(\Delta)\right]. \tag{1}$$

Note that the dual $c^{\vee}=c(\mathrm{d}/\mathrm{d}\mathbf{x})$ of a linear form $c=c(\mathbf{x})$ acts on the linear stress space, and therefore defines an action $\mathbb{K}[\Psi]\times S(\Psi)\to S(\Psi)$. For a a relative simplicial complex is a pair of simplicial complexes $\Psi=(\Delta,\Gamma)$, $\Gamma\subset\Delta$. If $\Psi=(\Delta,\Gamma)$ is a relative simplicial complex, it is not hard to modify this definition to obtain the correct picture: we simply define $S(\Psi)=\langle I_\Delta\rangle^{\perp}/\mathbb{K}[\Gamma]$.

Proposition 3.1 (cf. [Lee96]). *Assume the $(d-1)$-skeleton of a simplicial d-complex Ψ is properly embedded in $\mathbb{K}^d$. Then*

$$\rho: S(\Psi)\ \longrightarrow\ \mathrm{M}(\Psi),$$

the map restricting a stress to its squarefree terms, is injective (and therefore an isomorphism).

3.1. Consider Δ a geometric simplicial complex of dimension $d-1$ in $\mathbb{K}^n$. We say that Δ is satisfies **partition of unity (in degree k)** if the map

$$A^k(\Delta;\Theta_\Delta)\ \longrightarrow\ \bigoplus_{v\in\Delta^{(0)}}A^k(\mathrm{st}_v\Delta;\Theta_\Delta) \tag{2}$$

is an injection (where $\Delta^{(k)}$ shall henceforth denote the collection of k-faces of a simplicial complex Δ). In other words, Δ satisfies partition of unity in degree k if and only if every degree k element in $S_k(\Delta;\Theta_\Delta)$, can be written as a sum of stresses supported in stars of vertices.

4. Application I: Poincaré duality

4.1. The relation of partition of unity to Poincaré duality and the level property is made clear by the following theorem.

Theorem 4.1. *Let Δ denote a proper geometric simplicial $(d-1)$-complex in $\mathbb{K}^n$.*

(1) If $A^*(\Delta;\Theta_\Delta)$ is a level ring of dimension zero, then it satisfies partition of unity for every degree $k<d$.

(2) In particular, if $A^*(\Delta;\Theta_\Delta)$ satisfies partition of unity for every degree $k<d$ and Δ is a locally level, then $A^*(\Delta;\Theta_\Delta)$ is level.

Here, a complex Δ is **locally level** if $A(\mathrm{st}_v\Delta, \partial\mathrm{st}_v\Delta)$ is level and $A^d(\mathrm{st}_v\Delta, \partial\mathrm{st}_v\Delta) \to A^d(\Delta)$ is an injection for every vertex v.

Proof of Theorem 4.1. The first claim is clear. For the second claim, we consider the surjection

$$\bigoplus_{v\in\Delta} A^k(\mathrm{st}_v\Delta, \partial\mathrm{st}_v\Delta; \Theta_\Delta) \twoheadrightarrow A^k(\Delta; \Theta_\Delta)$$

and the pairings

$$A^k(\mathrm{st}_v\Delta, \partial\mathrm{st}_v\Delta; \Theta_\Delta) \times S_d(\mathrm{st}_v\Delta, \partial\mathrm{st}_v\Delta; \Theta_\Delta) \longrightarrow S_{d-k}(\mathrm{st}_v\Delta; \Theta_\Delta), \quad \forall v \text{ vertices in } \Delta.$$

By assumption, these pairings are surjections for all vertices v. The claim follows by partition of unity. □

4.2. The notion of stress spaces motivates us to consider the notion of a **minimal cycle**: Consider a simplicial $(d-1)$-complex Γ in a vector space $\mathbb{K}^n$, where $\mathbb{K}$ is some infinite field. We call Γ is a minimal cycle if $A^*(\Gamma)_d \cong \mathbb{K}$. If, in addition, for every face σ of Γ, the natural inclusion map

$$A^*(\mathrm{st}^\circ_\sigma\Gamma; \Theta_\Gamma)_d \longrightarrow A^*(\Gamma; \Theta_\Gamma)_d$$

is an isomorphism, we call Γ a **locally minimal cycle**.

In terms of stress spaces, a minimal cycle is a geometric simplicial $(d-1)$-complex Γ with $S_d(\Gamma; \Theta_\Gamma) = 1$. Moreover Γ is locally minimal if every open star has a unique d-stress that is obtained as the restriction of the global unique d-stress.

4.3. Clearly, simplicial polytopes are minimal cycles, and so are Bergman complexes. Moreover, their cohomology rings both satisfy Poincaré duality, as shown by Gräbe and the authors respectively. This motivates us to find a common explanation.

Theorem 4.2. *The Stanley–Reisner ring of a minimal, locally Gorenstein $(d-1)$-dimensional cycle Δ in $\mathbb{K}^n$ is Gorenstein.*

Corollary 4.3. *The Stanley–Reisner ring of a locally minimal and combinatorially Cohen–Macaulay cycle Δ in $\mathbb{K}^n$ is Gorenstein.*

Corollary 4.4 (cf. [Grä84])**.** *The reduced Stanley–Reisner ring of a simplicial homology sphere is Gorenstein.*

Corollary 4.5 (cf. [AHK15])**.** *The cohomology ring of the Bergman complex of a matroid is a Poincaré duality algebra.*

Proof. The Bergman complex is clearly a minimal cycle, as the unit weight is the unique Minkowski weight on the top-dimensional faces. Moreover, the Bergman complex is Cohen–Macaulay following Folkman [Fol66] and Björner [Bjö80]. □

4.4. To start, we need to understand the relation of partition of unity to the Cohen–Macaulay property.

Lemma 4.6. *Consider a proper geometric Cohen–Macaulay $(d-1)$-complex in $\mathbb{K}^d$. Then, for every $k<d$, we have a surjection*

$$\bigoplus_{v\in\Delta^{(0)}} A^k(\mathrm{st}_v\Delta;\Theta_\Delta) \longrightarrow\!\!\!\!\!\rightarrow A^k(\Delta;\Theta_\Delta). \tag{3}$$

Proof. Let denote Θ denote the coordinate ideal of parameters for A^Δ. For the proof, we consider the Koszul complexes $K^\bullet_s := K^\bullet(\theta_{s+1})$ and the chain complexes $C^\bullet_s$ defined as

$$0 \longrightarrow A^*(\Delta;\Theta_s) \longrightarrow \bigoplus_{v\in\Delta^{(0)}} A^*(\mathrm{st}_v\Delta;\Theta_s) \longrightarrow \cdots \longrightarrow \bigoplus_{F\in\Delta^{(d-1)}} A^*(\mathrm{st}_F\Delta;\Theta_s) \longrightarrow 0.$$

Considering the double complex $K^\bullet_s\otimes C^\bullet_s$ and computing the total complex inductively gives

$$H^s(C^\bullet_d)\cong (H^{d-1}(\Delta))^{\binom{d}{s}}(+d-s),$$

where $(+j)$ denotes a shift in degree by j, so that the only nontrivial cohomology is in degree $d-s$, as desired. □

Remark 4.7. This Lemma seems to be somewhat folklore according to research more versed than us in combinatorial commutative algebra.

Proof of Theorem 4.2. Let $\Theta=(\theta_1,\cdots,\theta_n)$ denote a generic system of linear forms spanning the coordinate ideal of Δ.

Lemma 4.6, together with Theorem 4.1, implies that Δ is a level simplicial complex, so that the socle of $A^*(\widetilde{\Delta};\Theta_d)$ is concentrated in degree d. We now prove by induction on s, $n\geqslant s\geqslant d$. This is proved by induction, considering again the complexes $C^\bullet_s$ and $K^\bullet_s$ as in the proof of Lemma 4.6. The only caveat we have to keep in mind is that the multiplication maps

$$A^*(\Delta;\Theta_s)\xrightarrow{\times\theta_{s+1}} A^*(\Delta;\Theta_s)$$

(and analogously for vertex stars) are no longer injections. To replace this fact, note that

$$(\ker\,\theta_{s+1})_k \longrightarrow \bigoplus_{v \text{ vertex in } \Delta} A^*(\mathrm{st}_v\Delta;\Theta_s)$$

is an injection by the level property for all $k<d$. The claim follows by computation of the total complex as above. □

5. Orlik Solomon algebras of Fans and Poincaré duality

5.1. Let Δ denote any proper geometric simplicial complex in $\mathbb{K}^n$. For $p \leqslant n$, we associate to Δ the subgroup $\mathcal{F}_p\Delta$ of $\bigwedge^p \mathbb{K}^n$ generated by elements $\alpha_1 v_1 \wedge \alpha_2 v_2 \wedge \cdots \wedge \alpha_p v_p$, where $v_1, v_2, \cdots, v_p$ are vertices of Δ and $\alpha_i \in \mathbb{K}$. The groups $\mathcal{F}_p\Delta$ are called the *p-groups* following Itenberg, Milkalkin, Katzarkov and Zharkov [IKMZ], and play a key role in tropical Hodge theory.

Zharkov [Zha13] showed that the co-p groups $\mathcal{F}^*\Delta = \mathrm{Hom}(\mathcal{F}_*\Delta, \mathbb{K})$, seen as a graded vector space, admit a natural algebra structure, and are isomorphic to the projectivized Orlik–Solomon algebra for Δ the Bergman complex of a matroid M.

5.2. Motivated by certain special cases, Zharkov conjectured a relation between the Orlik–Solomon algebra of a matroid and the stress spaces of its Bergman complex. We prove such a relation here.

Theorem 5.1 (Partition of unity and Orlik Solomon algebras)**.** *Consider a locally minimal combinatorially Cohen–Macaulay cycle Δ in $\mathbb{K}^n$. Then for every k we have an exact sequence*

$$0 \longleftarrow A^k(\Delta;\Theta_\Delta) \longleftarrow \bigoplus_{v\in\Delta^{(0)}} A^k(\mathrm{st}^\circ_v\Delta;\Theta_\Delta) \longleftarrow \cdots$$
$$\cdots \longleftarrow \bigoplus_{\tau\in\Delta^{(k-1)}} A^k(\mathrm{st}^\circ_\tau\Delta;\Theta_\Delta) \leftarrow \bigoplus_{\sigma\in\Delta^{(k-1)}} A^k(\mathrm{st}^\circ_\sigma\Delta;\Theta_\Delta) \longleftarrow \mathcal{F}^k(\Delta) \longleftarrow 0$$

Proof. Exactness of

$$0 \longrightarrow S_k(\Delta;\Theta_\Delta) \longrightarrow \bigoplus_{v\in\Delta^{(0)}} S_k(\mathrm{st}^\circ_v\Delta;\Theta_\Delta) \longrightarrow \cdots \longrightarrow \bigoplus_{\sigma\in\Delta^{(k-1)}} S_k(\mathrm{st}^\circ_\sigma\Delta;\Theta_\Delta)$$

is trivial. For the last claim, we need to understand the relation of Δ to a simplicial complex we understand completely, so we embed Δ into a simplex Ω, to which we add a number of generic vertices so that $\mathcal{F}^k\Omega \cong \bigwedge^k \mathbb{K}^n$.

Then we have an exact sequence

$$\bigoplus_{\tau\in\Omega^{(k-2)}} A^k(\mathrm{st}^\circ_\tau\Omega;\Theta_\Omega) \longrightarrow \bigoplus_{\sigma\in\Omega^{(k-1)}} A^k(\mathrm{st}^\circ_\sigma\Omega;\Theta_\Omega) \longleftarrow \bigwedge^k \mathbb{K}^n \longleftarrow 0.$$

Now, consider an element in $\gamma \in \bigoplus_{\sigma\in\Delta^{(k-1)}} S_k(\mathrm{st}^\circ_\sigma\Delta;\Theta_\Delta)$. It remains to show that if γ is in the image of

$$\bigoplus_{\tau\in\Omega^{(k-2)}} A^k(\mathrm{st}^\circ_\tau\Delta;\Theta_\Omega) \xrightarrow{\tilde{\psi}} \bigoplus_{\sigma\in\Omega^{(k-1)}} A^k(\mathrm{st}^\circ_\sigma\Omega;\Theta_\Omega) \supset \bigoplus_{\sigma\in\Delta^{(k-1)}} S_k(\mathrm{st}^\circ_\sigma\Delta;\Theta_\Delta),$$

then it is in the image of

$$\bigoplus_{\tau\in\Delta^{(k-2)}} A^k(\mathrm{st}^\circ_\tau\Delta;\Theta_\Delta) \xrightarrow{\psi} \bigoplus_{\sigma\in\Delta^{(k-1)}} A^k(\mathrm{st}^\circ_\sigma\Delta;\Theta_\Delta).$$

Consider therefore an element α with $\widetilde{\psi}(\alpha)=\gamma$. Then α restricts to an element in $\overline{\alpha}\in S_k(\Omega,\Delta;\Theta_\Omega)$.

5.3. To prove the theorem, it therefore finally suffices to prove:

Lemma 5.2. *$\overline{\alpha}$ is the restriction of a stress in $S(\Omega;\Theta_\Omega)$*

To this end it suffices to prove that, with j the restriction map, the exactness of

$$0 \longrightarrow S_k(\Omega,\Delta;\Theta_\Omega)\Big/ jS(\Omega;\Theta_\Omega) \longrightarrow \cdots \longrightarrow \bigoplus_{\tau\in\Delta^{(k-2)}} S_k(\mathrm{st}^\circ_\tau\Omega,\mathrm{st}^\circ_\tau\Delta;\Theta_\Omega)\Big/ jS(\mathrm{st}^\circ_\tau\Omega;\Theta_\Omega).$$

This is proved analogously to the proof of partition of unity in Δ by a simple induction on the linear system Θ_Ω. □

6. Application II: Concentration properties for primitive cohomology

6.1. A simplical polytope in $\mathbb{R}^d$, identified with its boundary Δ is a $(d-1)$-dimensional simplicial complex, whose Stanley–Reisner ring is known to satisfy the hard Lefschetz theorem following the work of Saito and McMullen [McM93]. Classically, simplicial polytopes with vanishing primitive Betti numbers, i.e.

$$\dim\left(A(\Delta;\Theta_\Delta)\Big/\omega A(\Delta;\Theta_\Delta)\right)^k = 0 \text{ for some } k\leqslant\frac{d}{2}$$

are well understood following the work of Kalai [Kal87] and subsequently Murai–Nevo [MN13]. A problem, however, was to understand the upper bounds on primitive Betti numbers. Define, for an integer a, and postitive integers $i<j$

$$a^{\langle i,j\rangle} \ := \ \max\{\dim R^j : R \text{ commutative algebra generated in degree } 1, \dim R^i = a\}.$$

If we declare

$$g_k(\Delta) := \dim\left(A(\Delta;\Theta_\Delta)\Big/\omega A(\Delta;\Theta_\Delta)\right)^k$$

for $k\leqslant\frac{d}{2}$, then quite clearly $g_k^{\langle k,k+1\rangle}(\Delta)\geqslant g_{k+1}(\Delta)$. This condition, together with $g_k\geqslant 0$, this is known to be the only numerical condition to characterize simplicial polytopes of dimension d, see [BL80].

We are left to understand the following problem.

Open Problem 6.1. *How can we describe simplicial polytopes with $g_k^{\langle k,k+1\rangle}(\Delta)=g_{k+1}(\Delta)$?*

To give one possible answer to this problem, we need some simple notions:

- Let
$$a^{\langle i,1\rangle} = \min\{\dim R^1 : R \text{ commutative algebra generated in degree } 1, \dim R^i = a\}.$$
- The complex of k-cliques $\mathrm{Cl}_k(\Delta)$ of a complex Δ is the collection of simplices whose $(k-1)$-skeleton is in Δ.

6.2. We can therefore give one possible answer to Problem 6.1.

Theorem 6.2. *Given d and g_k with $k \leqslant \frac{d}{2}$, and consider a simplicial d-polytope with*
$$g_k^{\langle k,k+1\rangle}(\Delta) = g_{k+1}(\Delta).$$
Then there exists a set V of $g_k^{\langle i,1\rangle} + d + 1$ vertices of Δ such that the d-skeleton of
$$\mathrm{Cl}_{k+1}(\Delta \cup \mathrm{Cl}_0 V)$$
is Cohen–Macaulay.

Proof. Let Θ denote the coordinate ideal of Δ, which we may assume to be in general position, and let ω denote a Lefschetz element for $A(\Delta;\Theta)$. The squarefree monomials in $\mathbb{K}[\mathbf{x}]$ supported in Δ generate $A(\Delta;(\Theta,\omega))$, and therefore $S(\Delta;(\Theta,\omega))$ by Proposition 3.1.

Let $\mathbf{x}^\alpha$ be such a degree k monomial, and let v be one of its vertices. Then $x_v\mathbf{x}^\alpha \neq 0$ in $A(\Delta;\Theta)$ by assumption, so that the k-stress γ in $S(\Delta;(\Theta,\omega))$ is supported in the star of v. Since the image of $x_v\mathbf{x}^\alpha$ in $S(\mathrm{st}_v^\circ\Delta;(\Theta,\omega))$ is minimal, we conclude that the faces generating γ lie in a common k-dimensional clique. By the upper bound theorem for Cohen–Macaulay complexes, it follows that all nontrivial primitive k-stresses are supported on at most $g_k^{\langle i,1\rangle} + d + 1$ vertices V of Δ.

Consider now
$$\widetilde{\Delta} \ := \ \Delta \cup \mathrm{C}, \text{ where } \mathrm{C} \ := \ \mathrm{Cl}_0 V.$$
We have $(\widetilde{\Delta}, \mathrm{C}) \ = \ (\Delta, \Delta_V)$, so that we have surjections
$$\omega : S_{k+1}(\widetilde{\Delta};\Theta) \ \longrightarrow \ S_k(\widetilde{\Delta},\mathrm{C};\Theta)\ S_{k-1}(\widetilde{\Delta},\mathrm{C};\Theta)$$
It follows as in [Adi15, Theorem 4.2] that the d-skeleton $\mathrm{Cl}_{k+1}\widetilde{\Delta}$ is Cohen-Macaulay. □

An i-stacking is the direct sum with a simplex, with identification along a faces of codimension at most $i-1$ (cf. [McM04]). With this notion, it is immediate to see that Theorem 6.2 generalizes the "generalized lower bound theorem" in polytope theory, cf. [MN13].

Corollary 6.3. *If, in the situation of Theorem 6.2, we have*

$$(g_k^{\langle k,1\rangle})^{\langle 1,k\rangle} \;=\; g_k,$$

then Δ is obtained from a neighborly polytope by $(k+1)$-stackings.

6.3. To quantify this theorem, we again use partition of unity to characterize what happens with the remaining generators.

Proposition 6.4. *Let Δ denote a simplicial d-polytope as above, let Θ denote its coordinate ideal and let ω denote a Lefschetz element for $A(\Delta;\Theta)$. Then*

$$\bigoplus_{v\in\Delta^{(0)}} A^k(\mathrm{st}_v\Delta;(\Theta,\omega)) \;\longrightarrow\!\!\!\!\rightarrow\; A^k(\Delta;(\Theta,\omega))$$

for all $k<\frac{d}{2}$.

The proof is analogous to the previous instances of partition of unity. Compare this also to the quantative lower bound theorem (see [Adi15, Theorem 3.1]) Let $\mathrm{N}_v(\delta)\Delta$ to denote the ball of radius δ around a vertex v in a simplicial complex Δ, measured in edge-distance. Combining Theorem 6.2 and Proposition 6.4, we obtain:

Theorem 6.5. *Given d and g_k with $k\leqslant\frac{d}{2}$, and let $\varepsilon\geqslant 0$ be a nonnegative integer. Then there exists $\delta=\delta(\varepsilon)$ and $\delta'=\delta'(\varepsilon)$ such that every d-polytope with*

$$g_k^{\langle k,k+1\rangle}(\Delta)-g_{k+1}(\Delta)<\varepsilon.$$

Then there exists a set V of at most δ' vertices of Δ such that the d-skeleton of

$$\mathrm{Cl}_{k+1}\left(\Delta\cup\bigcup_{v\in V}\mathrm{Cl}_0\,\mathrm{N}_v(\delta)\Delta\right)$$

is Cohen–Macaulay.

In other words, outside some set of vertices V that can be grouped into sets of small diameter, the polytope is $(k+1)$-stacked. We conclude the answer to a conjecture of Kalai [Kal94].

Corollary 6.6. *If (P_n) is a sequence of polytopes approximating a C^1 smooth convex body K in $\mathbb{R}^d$ in the Hausdorff topology. Then*

$$g_k^{\langle k,k+1\rangle}(P_n)-g_{k+1}(P_n)\to\infty.$$

as $n\to\infty$ for all $k\leqslant\frac{d}{2}$.

Proof. This follows as in [ANS15]. □

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